Distributed Learning Dynamics
Convergence in the Routing Game and Beyond

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Learning dynamics in the routing game

- Routing games model congestion on networks. Concise and elegant theory.
- Nash equilibrium quantifies efficiency of network in steady state.
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System does not operate at equilibrium. Beyond equilibria, we need to understand decision dynamics (learning).
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System **does not operate at equilibrium.** Beyond equilibria, we need to understand **decision dynamics (learning).**
- A realistic model for decision dynamics is essential for prediction, optimal control.
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System does not operate at equilibrium. Beyond equilibria, we need to understand decision dynamics (learning).

- A realistic model for decision dynamics is essential for prediction, optimal control.
Desiderata

Learning dynamics should be

- **Realistic** in terms of information requirements, computational complexity.
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- **Consistent** with the full information Nash equilibrium.

\[ x^{(t)} \to x^* \]

Convergence rates?
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\[ x(t) \to x^* \]

Convergence rates?

- **Robust** to stochastic perturbations.
  - Observation noise
  - (Bandit feedback)
Outline

1 Introduction

2 Convergence of agent dynamics

3 Routing Examples

4 Related problems
Outline

1. Introduction
2. Convergence of agent dynamics
3. Routing Examples
4. Related problems
Decision maker $k$ faces a sequential decision problem
At iteration $t$

1. chooses probability distribution $x^{(t)}_{A_k}$ over action set $A_k$
2. discovers a loss function $\ell_{A_k}^{(t)} : A_k \rightarrow [0, 1]$
3. updates distribution

\[
x_{A_k}^{(t+1)} = u\left(x_{A_k}^{(t)}, \ell_{A_k}^{(t)}\right)
\]

**Figure:** Sequential decision problem.
Interaction of $K$ decision makers

Decision maker $k$ faces a sequential decision problem
At iteration $t$
(1) chooses probability distribution $x_{A_k}^{(t)}$ over action set $A_k$
(2) discovers a loss function $\ell_{A_k}^{(t)} : A_k \rightarrow [0, 1]$
(3) updates distribution

Learning algorithm

$$x_{A_k}^{(t+1)} = u\left(x_{A_k}^{(t)}, \ell_{A_k}^{(t)}\right)$$

Environment

Other agents

Agent $k$

Outcome

$$\ell_{A_k}(x_{A_1}^{(t)}, \ldots, x_{A_K}^{(t)})$$

Figure: Sequential decision problem.

Loss of agent $k$ affected by strategies of other agents.
Does not know this function, only observes its value.
Write $x^{(t)} = (x_{A_1}^{(t)}, \ldots, x_{A_K}^{(t)})$. 
Examples of decentralized decision makers

Routing game

- Player drives from source to destination node
- Chooses path from $A_k$
- Mass of players on each edge determines cost on that edge.

**Figure:** Routing game
Examples of decentralized decision makers

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Figure: Routing game
Online Learning Model

1: for $t \in \mathbb{N}$ do
2: Play $p \sim x_A^{(t)}$
3: Discover $\ell_A^{(t)}$
4: Update

$$x_{A_k}^{(t+1)} = u_k \left( x_{A_k}^{(t)}, \ell_{A_k}^{(t)} \right)$$

5: end for
Online learning model

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Online Learning Model

1: for $t \in \mathbb{N}$ do
2: \hspace{1em} Play $p \sim x_{A_k}^{(t)}$
3: \hspace{1em} Discover $\ell_{A_k}^{(t)}$
4: \hspace{1em} Update $x_{A_k}^{(t+1)} = u_k \left( x_{A_k}^{(t)}, \ell_{A_k}^{(t)} \right)$
5: end for

$x_{A_1}^{(t)} \in \Delta^{A_1}$  Sample $p \sim x_{A_1}^{(t)}$  Discover $\ell_{A_1}^{(t)}$  Update $x_{A_1}^{(t+1)}$
Online learning model

Online Learning Model

1: \textbf{for} \( t \in \mathbb{N} \) \textbf{do}
2: \hspace{1cm} Play \( p \sim x_{A_k}^{(t)} \)
3: \hspace{1cm} Discover \( \ell_{A_k}^{(t)} \)
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\begin{align*}
x_{A_1}^{(t)} & \in \Delta^{A_1} \\
\text{Sample} \ p & \sim x_{A_1}^{(t)} \\
\text{Discover} \ \ell_{A_1}^{(t)} \\
\text{Update} \ x_{A_1}^{(t+1)}
\end{align*}

Main problem

Define class of dynamics \( C \) such that

\[ u_k \in C \ \forall k \Rightarrow x^{(t)} \rightarrow \chi^* \]
A brief review

Continuous-time:

Discrete time:

- Hannan consistency: [10]
- Hedge algorithm for two-player games: [9]
- Regret based algorithms: [11]
- Online learning in games: [7]
- Potential games: [19]

Specifically to the routing game

- No-regret dynamics [4], [14]


This talk

- Overview of some techniques for design and analysis of learning dynamics.
- Formulated for routing games. Extend to other classes of games.
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Nash equilibria, and the Rosenthal potential

Write
\[ x = (x_{A_1}, \ldots, x_{A_K}) \in \Delta^{A_1} \times \cdots \times \Delta^{A_K} \]
\[ \ell(x) = (\ell_{A_1}(x), \ldots, \ell_{A_K}(x)) \]

Nash equilibrium

\( x^* \) is a Nash equilibrium if
\[ \langle \ell(x^*), x - x^* \rangle \geq 0 \ \forall x \iff \forall k, \forall x_{A_k}, \langle \ell_{A_k}(x^*), x_{A_k} - x_{A_k}^* \rangle \geq 0 \]

In words, for all \( k \), paths in the support of \( x_{A_k}^* \) have minimal loss.
Nash equilibria, and the Rosenthal potential

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In words, for all \( k \), paths in the support of \( x^*_{A_k} \) have minimal loss.

**Figure:** Population distributions and noisy path losses
Nash equilibria, and the Rosenthal potential

Rosenthal potential

\[ \exists f \text{ convex such that} \]
\[ \nabla f(x) = \ell(x) \]

Then the set of Nash equilibria is

\[ X^* = \arg \min_{x \in \Delta^{A_1} \times \ldots \times \Delta^{A_K}} f(x) \]
Nash equilibria, and the Rosenthal potential

Rosenthal potential

\[ \exists f \text{ convex such that} \]
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Then the set of Nash equilibria is
\[ \mathcal{X}^* = \arg\min_{x \in \Delta A_1 \times \cdots \times \Delta A_K} f(x) \]

Nash condition \iff first order optimality
\[ \forall x, \langle \ell(x^*), x - x^* \rangle \geq 0 \quad \forall x, \langle \nabla f(x^*), x - x^* \rangle \geq 0 \]
Regret analysis

Technique 1: Regret analysis
Regret analysis

Cumulative regret

\[ R_{A_k}^{(t)} = \sup_{x_{A_k} \in \Delta A_k} \sum_{\tau \leq t} \left\langle x_{A_k}^{(t)} - x_{A_k}, \ell_{A_k}(x^{(t)}) \right\rangle \]

“Online” optimality condition. Sublinear if \( \limsup_t \frac{R_{A_k}^{(t)}}{t} \leq 0 \).

Convergence of averages

\[ \left[ \forall k, R_{A_k}^{(t)} \text{ is sublinear} \right] \Rightarrow \bar{x}^{(t)} \to x^* \]

\[ \bar{x}^{(t)} = \frac{1}{t} \sum_{\tau=1}^{t} x^{(\tau)}. \]
Convergence of $\bar{x}(t)$ Vs. convergence of $x(t)$

Routing game example

**Figure**: Population distributions
Convergence of $\bar{x}(t)$ Vs. convergence of $x(t)$

Routing game example

Path losses $\ell_{A_k}(x(t))$

Population 1

Population 2

Figure: Path losses
From convergence of $\bar{x}(t)$ to convergence of $x(t)$

Sufficient condition for $(x(t))_t \to x^*$

$f(x(t))$ eventually decreasing

$\downarrow$

$f(x(t)) \to f^*$

$\downarrow$

$x(t) \to x^*$
Technique 2: Stochastic approximation
Stochastic approximation

Idea:

- View the learning dynamics as a discretization of an ODE.
- Study convergence of ODE.
- Relate convergence of discrete algorithm to convergence of ODE.

Figure: Underlying continuous time
Example: the Hedge algorithm

Hedge algorithm

Update the distribution according to observed loss

\[ x_a^{(t+1)} \propto x_a^{(t)} e^{-\eta_t k_a^{(t)}} \]


Example: the Hedge algorithm

**Hedge algorithm**

Update the distribution according to observed loss

\[ x_a^{(t+1)} \propto x_a^{(t)} e^{-\eta_t \ell_a^{(t)}} \]

Also known as

- Exponentially weighted average forecaster [7].

---


Example: the Hedge algorithm

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Example: the Hedge algorithm

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Update the distribution according to observed loss

\[ x_{a}^{(t+1)} \propto x_{a}^{(t)} e^{-\eta_{t}^{k} \ell_{a}^{(t)}} \]

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- Exponentially weighted average forecaster [7].
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Example: the Hedge algorithm

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Update the distribution according to observed loss

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Also known as
- Exponentially weighted average forecaster [7].
- Multiplicative weights update [1].
- Exponentiated gradient descent [13].
- Entropic descent [2].
- Log-linear learning [5], [18]

The replicator ODE

In Hedge $x^{(t+1)}_p \propto x^{(t)}_p e^{-\eta^k_t \ell^{(t)}_p}$, take $\eta_t \to 0$.

Replicator equation [27]

$$\forall a \in A_k, \frac{dx_a}{dt} = x_a (\langle \ell_{A_k}(x), x_{A_k} \rangle - \ell_a(x))$$  \hspace{1cm} (1)


The replicator ODE

In Hedge $x^{(t+1)}_p \propto x^{(t)}_p e^{-\eta^k t p^{(t)}}$, take $\eta_t \rightarrow 0$.

**Replicator equation [27]**

$$\forall a \in A_k, \frac{dx_a}{dt} = x_a (\langle \ell_A(x), x_A \rangle - \ell_a(x))$$ (1)

**Theorem: [8]**

Every solution of the ODE (1) converges to the set of its stationary points.

---


AREP dynamics: Approximate REPlicator

Discretization of the continuous-time replicator dynamics

\[ x_a^{(t+1)} - x_a^{(t)} = \eta_t x_a^{(t)} \left( \langle \ell_{A_k}(x^{(t)}), x_{A_k}^{(t)} \rangle - \ell_a(x^{(t)}) \right) + \eta_t U_a^{(t+1)} \]

- \((U^{(t)})_{t \geq 1}\) perturbations that satisfy for all \(T > 0\),

\[ \lim_{\tau_1 \to \infty} \max_{\tau_2 : \sum_{t=\tau_1}^{\tau_2} \eta_t < T} \left\| \sum_{t=\tau_1}^{\tau_2} \eta_t U^{(t+1)} \right\| = 0 \]

- \(\eta_t\) discretization time steps.

(a sufficient condition is that \(\exists q \geq 2: \sup_{T} \mathbb{E} \|U^{(T)}\|^q < \infty\) and \(\sum_{T} \eta_T^{1+\frac{q}{2}} M_T \infty\))

Convergence to Nash equilibria

**Theorem [16]**

Under AREP updates, if $\eta_t \downarrow 0$ and $\sum \eta_t = \infty$, then

$$x(t) \rightarrow \mathcal{X}^*$$

- Affine interpolation of $x(t)$ is an asymptotic pseudo trajectory.
- Use $f$ as a Lyapunov function.  

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Convergence to Nash equilibria

Theorem [16]
Under AREP updates, if $\eta_t \downarrow 0$ and $\sum \eta_t = \infty$, then

$$x(t) \to X^*$$

- Affine interpolation of $x(t)$ is an asymptotic pseudo trajectory.

Use $f$ as a Lyapunov function. proof details

However, No convergence rates.

Stochastic convex optimization

Technique 3: (Stochastic) convex optimization
Stochastic convex optimization

Idea:
- View the learning dynamics as a **distributed algorithm to minimize** $f$.
- (More generally: distributed algorithm to find zero of a monotone operator).
Stochastic convex optimization

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- Allows us to analyze convergence rates.
Stochastic convex optimization

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- View the learning dynamics as a distributed algorithm to minimize $f$.
- (More generally: distributed algorithm to find zero of a monotone operator).
- Allows us to analyze convergence rates.

Here:
Class of distributed optimization methods: stochastic mirror descent.
Stochastic Mirror Descent

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in \mathcal{X} \subset \mathbb{R}^d
\end{align*}
\]

convex function

convex, compact set

---


Stochastic Mirror Descent

\[
\begin{align*}
\text{minimize} & \quad f(x) & & \text{convex function} \\
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\end{align*}
\]

**Algorithm 2** MD Method with learning rates \((\eta_t)\)

1. **for** \(t \in \mathbb{N} \) **do**
2. observe \(\ell(t) \in \partial f(x(t))\)
3. \(x(t+1) = \arg \min_{x \in \mathcal{X}} \left\langle \ell(t), x \right\rangle + \frac{1}{\eta_t} D_\psi(x, x(t))\)
4. **end for**

- \(\eta_t\): learning rate
- \(D_\psi\): Bregman divergence


**Stochastic Mirror Descent**

minimize \( f(x) \)  
subject to \( x \in \mathcal{X} \subset \mathbb{R}^d \)  

**Algorithm 2** MD Method with learning rates \((\eta_t)\)

1: \textbf{for} \( t \in \mathbb{N} \) \textbf{do}
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3: \hspace{1em} \( x^{(t+1)}_{A_k} = \arg \min_{x \in \mathcal{X}_{A_k}} \langle \ell^{(t)}_{A_k}, x \rangle + \frac{1}{\eta_t} D_{\psi_k}(x, x^{(t)}_{A_k}) \)
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Stochastic Mirror Descent

minimize \( f(x) \)  \hspace{1cm} \text{convex function}

subject to \( x \in \mathcal{X} \subset \mathbb{R}^d \)  \hspace{1cm} \text{convex, compact set}

**Algorithm 2 SMD Method with learning rates (\( \eta_t \))**

1: for \( t \in \mathbb{N} \) do
2: \hspace{1cm} observe \( \hat{\ell}_A^{(t)} \) with \( \mathbb{E} \left[ \hat{\ell}_A^{(t)} | \mathcal{F}_{t-1} \right] \in \partial A_k f(x^{(t)}) \)
3: \hspace{1cm} \( x^{(t+1)}_{A_k} = \arg \min_{x \in \mathcal{X}_A k } \left\langle \hat{\ell}_A^{(t)}, x \right\rangle + \frac{1}{\eta_t} D_{\psi_k} (x, x^{(t)}_{A_k}) \)
4: end for

- \( \eta_t \): learning rate
- \( D_{\psi} \): Bregman divergence


Under mirror descent, $f(\bar{x}(t)) \to f^*$. 

**A true descent [17]**

If $\nabla f$ is Lipschitz, and $\eta_t \downarrow 0$, then eventually,

$$f(x^{(t+1)}) \leq f(x^{(t)})$$

Then under mirror descent with $\sum \eta_t = \infty$,

$$f(x^{(t)}) - f^* \leq O \left( \frac{\sum_{\tau \leq t} \eta_{\tau}}{t} + \frac{1}{t\eta_t} + \frac{1}{t} \right)$$

**Figure:** Mirror Descent iteration with decreasing $\eta_t$

---

Deterministic version: a true descent

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In European Control Conference (ECC), 2015
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Figure: Mirror Descent iteration with decreasing \( \eta_t \)

In European Control Conference (ECC), 2015
Stochastic version

Know: $\mathbb{E}[f(\bar{x}(t))] \to f^*$ [20] (more general averaging)

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<tr>
<th>$f$</th>
<th>$\eta_t$</th>
<th>Convergence</th>
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<td>Weakly convex</td>
<td>$\frac{\theta_k}{t^{\alpha_k}}$, $\alpha_k \in (0, 1)$</td>
<td>$\mathbb{E}\left[f(x^{(t)}) - f^*\right] = O\left(\sum_k \log t \min(\alpha_k, 1 - \alpha_k)\right)$</td>
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<tr>
<td>Strongly convex</td>
<td>$\frac{\theta_k}{\ell_f t^{\alpha_k}}$, $\alpha_k \in (0, 1)$</td>
<td>$\mathbb{E}\left[D_{\psi}(x^*, x^{(t)})\right] = O(\sum_k t^{-\alpha_k})$</td>
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Figure: SMD convergence rates [15]

General algorithm: applications beyond distributed learning models. E.g. large scale machine learning. ▶ More details


Convergence

\[ d_\tau = D_\psi(\mathcal{X}^*, x^{(\tau)}) \].

**Main ingredient**

\[
\mathbb{E}[d_{\tau+1}|\mathcal{F}_{\tau-1}] \leq d_\tau - \eta_\tau (f(x^{(\tau)}) - f^*) + \frac{\eta^2_\tau}{2\mu} \mathbb{E}\left[\|\hat{\ell}^{(\tau)}\|^2_*|\mathcal{F}_{\tau-1}\right]
\]

---

[22] H. Robbins and D. Siegmund. *A convergence theorem for non negative almost supermartingales and some applications.*  
*Optimizing Methods in Statistics*, 1971

1998
Convergence

\[ d_{\tau} = D_\psi(\mathcal{X}^*, x^{(\tau)}) \]

**Main ingredient**

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\mathbb{E}[d_{\tau+1}|\mathcal{F}_{\tau-1}] \leq d_{\tau} - \eta_\tau (f(x^{(\tau)}) - f^*) + \frac{\eta_\tau^2}{2\mu} \mathbb{E}[\|\hat{\ell}^{(\tau)}\|_*^2|\mathcal{F}_{\tau-1}]
\]

From here,

- Can show a.s. convergence \( x^{(t)} \rightarrow \mathcal{X}^* \) if \( \sum \eta_t = \infty \) and \( \sum \eta_t^2 < \infty \)

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From here,

- Can show a.s. convergence \(x^{(t)} \rightarrow \mathcal{X}^*\) if \(\sum \eta_t = \infty\) and \(\sum \eta_t^2 < \infty\)
- \(d_\tau\) is an almost super martingale [22], [6]

---


Convergence

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\mathbb{E}[d_{\tau+1}|\mathcal{F}_{\tau-1}] \leq d_\tau - \eta_\tau (f(x^{(\tau)}) - f^*) + \frac{\eta_\tau^2}{2\mu} \mathbb{E}[\|\hat{\ell}(\tau)\|_*^2|\mathcal{F}_{\tau-1}]
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From here,

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  - \( d_\tau \) is an almost super martingale [22], [6]

Deterministic version: \( d_{\tau+1} \leq d_\tau - a_\tau + b_\tau, \sum b_\tau < \infty \).

---


Convergence

- To show convergence $\mathbb{E} \left[ f(x^{(t)}) \right] \rightarrow f^*$, generalize the technique of Shamir et al. [25] (for SGD, $\alpha = \frac{1}{2}$).

Convergence of Distributed Stochastic Mirror Descent

For $\eta_t^k = \frac{\theta_k}{t^{\alpha_k}}$, $\alpha_k \in (0, 1)$,

$$\mathbb{E} \left[ f(x^{(t)}) \right] - f^* = O \left( \sum_k \frac{\log t}{t^{\min(\alpha_k, 1-\alpha_k)}} \right)$$

Non-smooth, non-strongly convex.


Summery

- Regret analysis: convergence of $\bar{x}^{(t)}$
- Stochastic approximation: almost sure convergence of $x^{(t)}$
- Stochastic convex optimization: almost sure convergence,
  $\mathbb{E} \left[ f(x^{(t)}) \right] \to f^*$, $\mathbb{E} \left[ D_\psi(x^*, x^{(t)}) \right] \to 0$, convergence rates.
Outline

1. Introduction
2. Convergence of agent dynamics
3. Routing Examples
4. Related problems
Application to the routing game

- Centered Gaussian noise on edges.
- Population 1: Hedge with $\eta_t^1 = t^{-1}$
- Population 2: Hedge with $\eta_t^2 = t^{-1}$
Routing game with strongly convex potential

**Mass distributions** $x_{A_k}^{(t)}$

**Path losses** $\ell_{A_k}(x^{(t)})$

Figure: Population distributions and noisy path losses
Routing game with strongly convex potential

For $\eta^k_t = \frac{\theta_k}{\ell_f t^\alpha_k}$, $\alpha_k \in (0, 1]$, $\mathbb{E} [D_{\psi}(x^*, x(t))] = O(\sum_k t^{-\alpha_k})$
Routing game with weakly convex potential

Figure: A weakly convex example.
Routing game with weakly convex potential

For $\frac{\theta_k}{t^{\alpha_k}}$, $\alpha_k \in (0, 1)$, $\mathbb{E} \left[ f(x(t)) \right] - f^* = O \left( \sum_k \frac{\log t}{t \min(\alpha_k, 1 - \alpha_k)} \right)$
Routing game with weakly convex potential

Figure: Potential values.

For $\frac{\theta_k}{t^{\alpha_k}}$, $\alpha_k \in (0, 1)$, $\mathbb{E}[f(x(t))] - f^* = O \left( \sum_k \frac{\log t}{t^{\min(\alpha_k, 1-\alpha_k)}} \right)$
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Figure: Potential values.

For $\theta_k \frac{1}{t^{\alpha_k}}$, $\alpha_k \in (0, 1)$, $\mathbb{E}[f(x(t))] - f^* = O\left(\sum_k \frac{\log t}{t^{\min(\alpha_k, 1-\alpha_k)}}\right)$
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A routing experiment

- Interface for the routing game.
- Used to collect sequence of decisions $\chi(t)$.

**Figure:** Interface for the routing game experiment.
Estimation of learning dynamics

Suppose we observe

- A sequence of player decisions \((x^{(t)})\)
- The corresponding sequence of losses \((\ell^{(t)})\)

Can we fit a model of player dynamics?
Estimation of learning dynamics

Suppose we observe

- A sequence of player decisions \( (x^{(t)}) \)
- The corresponding sequence of losses \( (\ell^{(t)}) \)

Can we fit a model of player dynamics?

Simple model: estimate the learning rate in the mirror descent model

\[
\hat{x}^{(t+1)}(\eta) = \arg \min_{x \in \Delta^{A_k}} \left\langle \ell^{(t)}, x \right\rangle + \frac{1}{\eta} D_{KL}(x, x^{(t)})
\]
Suppose we observe

- A sequence of player decisions \((x^{(t)})\)
- The corresponding sequence of losses \((\ell^{(t)})\)

Can we fit a model of player dynamics?

Simple model: estimate the learning rate in the mirror descent model

\[
\tilde{x}^{(t+1)}(\eta) = \arg\min_{x \in \Delta A_k} \left< \ell^{(t)}, x \right> + \frac{1}{\eta} D_{KL}(x, x^{(t)})
\]

Then \(d(\eta) = D_{KL}(x^{(t+1)}, \tilde{x}^{(t+1)}(\eta))\) is a convex function. Can minimize it to estimate \(\eta_k^{(t)}\).
Estimation of learning dynamics

Figure: Learning rate estimates using the entropy model.
Optimal routing with learning dynamics

Assume

- a central authority has control over a fraction of traffic: \( u(t) \)
- Rest of traffic follows learning dynamics: \( x(t) \)

\[
\begin{align*}
\text{minimize}_{u(1:T), x(1:T)} & \quad \sum_{t=1}^{T} J(x(t), u(t)) \\
\text{subject to} & \quad x(t+1) = u(x(t) + u(t), \ell(x(t) + u(t)))
\end{align*}
\]
Optimal routing with learning dynamics

Figure: Los Angeles highway network.
Optimal routing with learning dynamics

Figure: Average delay without control (dashed), with full control (solid), and different values of $\alpha$. 
Summary

- Simple model for distributed learning.
- Techniques for design / analysis of learning dynamics:
  - Regret analysis, stochastic approximation, stochastic optimization.
- Related problems not covered here: Infinite action sets, accelerated dynamics.
Introduction
Convergence of agent dynamics
Routing Examples
Related problems
References

Summary

**Figure:** Coupled sequential decision problems.

- Simple model for distributed learning.
- Techniques for design / analysis of learning dynamics: Regret analysis, stochastic approximation, stochastic optimization.
- Related problems not covered here: Infinite action sets, accelerated dynamics.
- Many brilliant visiting students / undergrads

Benjamin Drighès
Milena Suarez
Syrine Krichene
Kiet Lam

Thank you!

eecs.berkeley.edu/~walid/


References V


Continuous time model

Continuous-time learning model

\[ \dot{x}_{A_k}(t) = v_k \left( x^{(t)}_{A_k}, \ell_{A_k}(x^{(t)}) \right) \]

- Evolution in populations: [24]
- Convergence in potential games under dynamics which satisfy a positive correlation condition [23]
- Replicator dynamics for the congestion game [8] and in evolutionary game theory [27]
- No-regret dynamics for two player games [11]


Oscillating example

Path losses $\ell_{A_k}(x(t))$

- Population 1
  - Path $p_0 = (v_0, v_5, v_1)$
  - Path $p_1 = (v_0, v_4, v_5, v_1)$
  - Path $p_2 = (v_0, v_1)$

- Population 2
  - Path $p_0 = (v_0, v_5, v_1)$
  - Path $p_1 = (v_0, v_4, v_5, v_1)$
  - Path $p_2 = (v_0, v_1)$

Figure: Path losses
Oscillating example

Figure: Potentials
Oscillating example

Figure: Trajectories in the simplex
Regret $[10]$  

### Cumulative regret

$$R_{A_k}^{(t)} = \sup_{x_{A_k} \in \Delta A_k} \sum_{\tau \leq t} \left\langle x_{A_k}^{(t)} - x_{A_k}, \ell_{A_k}(x^{(t)}) \right\rangle$$

### Convergence of averages

$$\forall k, \limsup_{t} \frac{R_{A_k}^{(t)}}{t} \leq 0 \Rightarrow \bar{x}^{(t)} = \frac{1}{t} \sum_{\tau \leq t} x^{(\tau)} \to \mathcal{X}^*$$

By convexity of $f$,

$$f \left( \frac{1}{t} \sum_{\tau \leq t} x^{(\tau)} \right) - f(x) \leq \frac{1}{t} \sum_{\tau \leq t} f(x^{(\tau)}) - f(x)$$

$$\leq \frac{1}{t} \sum_{\tau \leq t} \left\langle \ell(x^{(t)}), x^{(t)} - x \right\rangle = \sum_{k=1}^{K} R_{A_k}^{(t)} \frac{1}{t}$$

---

*Contributions to the Theory of Games, 3:97–139, 1957*
AREP convergence proof

- Affine interpolation of $x^{(t)}$ is an asymptotic pseudo trajectory.

- The set of limit points of an APT is internally chain transitive ICT.
- If $\Gamma$ is compact invariant, and has a Lyapunov function $f$ with $\text{int} f(\Gamma) = \emptyset$, then $\forall L$ ICT, $\Gamma$, and $f$ is constant on $L$.
- In particular, $f$ is constant on $L(x^{(t)})$, so $f(x^{(t)})$ converges.
**Bregman Divergence**

**Strongly convex function** $\psi$

$$D_{\psi}(x, y) = \psi(x) - \psi(y) - \langle \nabla \psi(y), x - y \rangle$$

---

Bregman Divergence

Strongly convex function $\psi$

$$D_{\psi}(x, y) = \psi(x) - \psi(y) - \langle \nabla \psi(y), x - y \rangle$$

Example [2]: when $\mathcal{X} = \Delta^d$

- $\psi(x) = -H(x) = \sum_a x_a \ln x_a$
- $D_{\psi}(x, y) = D_{KL}(x, y) = \sum_a x_a \ln \frac{x_a}{y_a}$
- The MD update has closed form solution

$$x^{(t+1)} \propto x^{(t)}_a e^{-\eta t s_a^{(t)}}$$

A.k.a. Hedge algorithm, exponential weights.

Figure: KL divergence

A bounded entropic divergence

- $\mathcal{X} = \Delta$
- $D_{KL}(x, y) = \sum_{i=1}^{d} x_i \ln \frac{x_i}{y_i}$ is unbounded.
A bounded entropic divergence

\[ \mathcal{X} = \Delta \]

\[ D_{KL}(x, y) = \sum_{i=1}^{d} x_i \ln \frac{x_i}{y_i} \] is unbounded.

Define \[ D_{KL}^\epsilon(x, y) = \sum_{i=1}^{d} (x_i + \epsilon) \ln \frac{x_i + \epsilon}{y_i + \epsilon} \]

**Proposition**

- \( D_{KL}^\epsilon \) is \( \frac{1}{1+d\epsilon} \)-strongly convex w.r.t. \( \| \cdot \|_1 \)
- \( D_{KL}^\epsilon \) is bounded by \( (1 + d\epsilon) \ln \frac{1+\epsilon}{\epsilon} \).
Theorem: Convergence of DMD [17]

Suppose $f$ has $L$ Lipschitz gradient. Then under the MD class with $\eta_t \downarrow 0$ and $\sum \eta_t = \infty$,

$$f(x^{(t)}) - f^* = O \left( \frac{\sum_{\tau \leq t} \eta\tau}{t} + \frac{1}{\eta_t} + \frac{1}{t} \right)$$

$$\frac{1}{t} \sum_{\tau \leq t} f(x^{(t)}) - f^* \leq \sum_k \frac{L^2_k}{2\ell\psi_k} \sum_{\tau \leq t} \eta^k\tau + \frac{D_k}{\eta^k_t}$$

and

$$f(x^{(t)}) - f^* \leq \frac{1}{t} \sum_{\tau \leq t} f(x^{(t)}) - f^* + O \left( \frac{1}{t} \right)$$
Convergence in DSMD

Regret bound [15]

SMD method with \((\eta_t)\). \(\forall t_2 > t_1 \geq 0\) and \(\mathcal{F}_{t_1}\)-measurable \(x\),

\[
\sum_{\tau = t_1}^{t_2} \mathbb{E} \left[ \langle g(\tau), x(\tau) - x \rangle \right] \leq \frac{\mathbb{E} \left[ D_\psi(x, x(t_1)) \right]}{\eta_{t_1}} + D \left( \frac{1}{\eta_{t_2}} - \frac{1}{\eta_{t_1}} \right) + \frac{G}{2\ell_\psi} \sum_{\tau = t_1}^{t_2} \eta_{\tau}
\]

Strongly convex case:

\[
\mathbb{E}[D_\psi(x^*, x^{(t+1)})] \leq (1 - 2\ell_f \eta_t) \mathbb{E}[D_\psi(x^*, x^{(t)})] + \frac{G}{2\ell_\psi} \eta_t^2
\]
Convergence in DSMD

Weakly convex case:

**Theorem [15]**

Distributed SMD such that \( \eta_t^p = \frac{\theta_p}{t^{\alpha_p}} \) with \( \alpha_p \in (0, 1) \). Then

\[
\mathbb{E} \left[ f(x(t)) \right] - f(x^*) \leq \left( 1 + \sum_{i=1}^{t} \frac{1}{i} \right) \sum_{k \in A} \left( \frac{1}{t^{1-\alpha_k}} \frac{D}{\theta_k} + \frac{\theta_k G}{2\ell_\psi(1-\alpha_k)} \frac{1}{t^{\alpha_k}} \right)
\]

\[
= O \left( \frac{\log t}{t^{\min(\min_k \alpha_k, 1-\max_k \alpha_k)}} \right)
\]

Define \( S_i = \frac{1}{i+1} \sum_{t-i}^{t} \mathbb{E}[f(x(\tau))] \)

Show \( S_{i-1} \leq S_i + \left( \frac{D}{\theta} \frac{1}{t^{\alpha-1}} + \frac{\theta G}{2\ell_\psi(1-\alpha)} \frac{1}{t^{\alpha}} \right) \frac{1}{i} \)

Stochastic mirror descent in machine learning

Large scale learning:

$$\text{minimize}_x \sum_{i=1}^{N} f_i(x)$$

subject to $x \in \mathcal{X}$

$N$ very large. Gradient prohibitively expensive to compute exactly. Instead, compute

$$\hat{g}(x^{(t)}) = \sum_{i \in \mathcal{I}} \nabla f_i(x^{(t)})$$

with $\mathcal{I}$ random subset of $\{1, \ldots, N\}$. 
**Accelerated MD**

<table>
<thead>
<tr>
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<th>Gradient descent</th>
<th>mirror decent</th>
</tr>
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<td>(stochastic) weakly convex</td>
<td>$\frac{1}{\sqrt{t}}$</td>
<td>$\frac{1}{\sqrt{t}}$</td>
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<tr>
<td>(stochastic) strongly convex</td>
<td>$\frac{1}{t}$</td>
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</tr>
<tr>
<td>strongly convex, accelerated</td>
<td>$\frac{1}{t^2}$</td>
<td>?</td>
</tr>
</tbody>
</table>

**Figure:** Convergence rates

Nesterov’s accelerated method: adds a momentum term with $\alpha_t = \frac{t-1}{t+2}$

\[
\begin{align*}
x(t) &= y^{(t-1)} - \eta \nabla f(y^{(t-1)}) \\
y(t) &= x(t) + \alpha_t (x(t) - x^{(t-1)})
\end{align*}
\]
Accelerated MD

- A recent interpretation of Nesterov's accelerated method [26]: discretization of the ODE

\[
\ddot{x}(t) + \frac{3}{t} \dot{x}(t) + \nabla f(x(t)) = 0 \\
\dot{x}(0) = 0
\]


Accelerated MD

- A recent interpretation of Nesterov's accelerated method [26]: discretization of the ODE

\[
\ddot{x}(t) + \frac{3}{t} \dot{x}(t) + \nabla f(x(t)) = 0 \\
\dot{x}(0) = 0
\]

- Mirror descent was motivated by continuous-time dynamics [21]:
  Choose a Bregman divergence \( D_\psi(x(t), x^*) \).

\[
\dot{x}(t) = -\nabla f(\nabla \psi(x(t)))
\]

Then \( D_\psi(x(t), x^*) \) is a Lyapunov function for the dynamics.

---

In *NIPS*, 2014

Lyapunov function proof

\[ \frac{d}{dt} D_\psi(x(t), x^*) = \frac{d}{dt} \left( \psi(x(t)) - \psi(x^*) - \langle \nabla \psi(x^*), x(t) - x^* \rangle \right) \]

\[ = \left\langle \nabla \psi(x(t)) - \nabla \psi(x^*), \frac{d}{dt} x(t) \right\rangle \]

\[ = \left\langle \nabla \psi(x(t)) - \nabla \psi(x^*), -\nabla f \psi(x(t)) \right\rangle \]

\[ \leq 0 \]