The Hedge Algorithm on a Continuum

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Outline

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Online Learning over a finite set

A decision maker faces a sequential problem:

Online decision problem over a finite set $\{1, \ldots, N\}$.

for $t \in \mathbb{N}$ **do**
- Decision maker chooses distribution $x^{(t)}$ over $\{1, \ldots N\}$.
- A loss vector $\ell^{(t)} \in [0, M]^N$ is revealed.
- The decision maker incurs expected loss $\sum_{n=1}^{N} \ell_n^{(t)} x_n^{(t)} = \left\langle x^{(t)}, \ell^{(t)} \right\rangle$

end for

Applications

Applications

- Convergence of player dynamics in games (Blackwell [1], Hannan[5]) $\{1, \ldots, N\}$ is the set of actions.

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[1]David Blackwell. An analog of the minimax theorem for vector payoffs. *Pacific Journal of Mathematics*, 6(1):1–8, 1956

[5]James Hannan. Approximation to Bayes risk in repeated plays. *Contributions to the Theory of Games*, 3:97–139, 1957

[4]Thomas M. Cover. Universal portfolios. *Mathematical Finance*, 1(1):1–29, 1991

[2]Avrim Blum and Adam Kalai. Universal portfolios with and without transaction costs. *Machine Learning*, 35(3):193–205, 1999

Applications

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- Convergence of player dynamics in games (Blackwell [1], Hannan[5])
  $\{1, \ldots, N\}$ is the set of actions.
- Machine Learning
  $\{1, \ldots, N\}$ is the training set.

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Applications

Applications

- Convergence of player dynamics in games (Blackwell [1], Hannan[5])
  $\{1, \ldots, N\}$ is the set of actions.
- Machine Learning
  $\{1, \ldots, N\}$ is the training set.
- "Model-free" portfolio optimization (Cover [4], Blum [2])
  $\{1, \ldots, N\}$ is the set of stocks.
- Many others

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Learning on a continuum

"What if the action set is infinite?"

Problem 1 Online decision problem on S.

for $t \in \mathbb{N}$ **do**

    Decision maker chooses distribution $x^{(t)}$ over S.
    A loss function $\ell^{(t)} : S \to [0, M]$ is revealed.
    The decision maker incurs expected loss

$$\left\langle x^{(t)}, \ell^{(t)} \right\rangle = \int_S x^{(t)}(s)\ell^{(t)}(s)\lambda(ds) = \mathbb{E}_{s\sim x^{(t)}}[\ell^{(t)}(s)]$$

end for

Learning on a continuum

"What if the action set is infinite?"

Problem 2 Online decision problem on S.

1: **for** $t \in \mathbb{N}$ **do**
2: Decision maker chooses distribution $x^{(t)}$ over S.
3: A loss function $\ell^{(t)} : S \to [0, M]$ is revealed.
4: The decision maker incurs expected loss

$$\left\langle x^{(t)}, \ell^{(t)} \right\rangle = \int_S x^{(t)}(s)\ell^{(t)}(s)\lambda(ds) = \mathbb{E}_{s\sim x^{(t)}}[\ell^{(t)}(s)]$$

5: **end for**

Regret

$$R^{(T)}(x) = \sum_{t=1}^{T} \left\langle x^{(t)}, \ell^{(t)} \right\rangle - \left\langle x, \sum_{t=1}^{T} \ell^{(t)} \right\rangle$$

$$\sup_{(\ell^{(t)})} \sup_{x\in\Delta^N} R^{(T)}(x) = o(T)$$

Results

Variant of this problem: Online optimization on convex sets.

Assumptions on $\ell^{(t)}$	convex	α-exp-concave	uniformly L-Lipschitz
Assumptions on S	convex	convex	v-uniformly fat
Method	Gradient (Zinkevich [8])	Hedge, ONS, FTAL (Hazan et al. [6])	Hedge (This talk)
Learning rates	$1/\sqrt{t}$	α	$1/\sqrt{t}$
$R^{(t)}$	$\mathcal{O}(\sqrt{t})$	$\mathcal{O}(\log t)$	$\mathcal{O}(\sqrt{t \log t})$

Table: Some regret upper bounds for different classes of losses.

[8]Martin Zinkevich. Online convex programming and generalized infinitesimal gradient ascent.
In *ICML*, pages 928–936, 2003

[6]Elad Hazan, Amit Agarwal, and Satyen Kale. Logarithmic regret algorithms for online convex optimization.
Machine Learning, 69(2-3):169–192, 2007

Outline

Hedge on a finite set

Hedge algorithm with learning rates (η_t).

for $t \in \mathbb{N}$ **do**
  Play $x^{(t)}$
  Reveal $\ell^{(t)} \in [0, M]^N$, call $L^{(t)} = \sum_{\tau=1}^{t} \ell^{(\tau)}$
  Update

$$x_n^{(t+1)} \propto e^{-\eta_{t+1} L_n^{(t)}}$$

end for

One interpretation: instance of the dual averaging method [7]

$$x^{(t+1)} \in \arg\min_{x \in \Delta^N} \left\langle L^{(t)}, x \right\rangle + \frac{1}{\eta_{t+1}} \psi(x)$$

with $\psi(x) = \sum_{n=1}^{N} x_n \ln x_n$.

[7]Yurii Nesterov. Primal-dual subgradient methods for convex problems. *Mathematical Programming*, 120(1):221–259, 2009

Hedge on a finite set

Basic Regret Bound

For all $x \in \Delta^N$,

$$R^{(T)}(x) \leq \frac{M^2}{2} \sum_{\tau=1}^{t} \eta_{\tau+1} + \frac{\psi(x)}{\eta_{t+1}}$$

Take $\eta_t = \theta t^{-\frac{1}{2}}$, then $\sum_1^t \eta_\tau = O(\sqrt{t})$ and $\frac{1}{t} = O(\sqrt{t})$
It suffices to bound ψ on Δ^N.
When $\psi(x) = \sum_i x_i \ln x_i$, $\psi(x) \leq \ln N$ on Δ^N. So

$$\sup_{x \in \Delta^N} R^{(T)}(x) \leq \left(\frac{M^2 \theta}{2} + \frac{\ln N}{\theta} \right) \sqrt{T}$$

Hedge on a continuum

Hedge on S with learning rates (η_t).

for $t \in \mathbb{N}$ **do**
  Play $\sim x^{(t)}$
  Reveal $\ell^{(t)} : S \to [0, M]$
  Update

$$x^{(t+1)}(s) \propto x^{(0)}(s) e^{-\eta_{t+1} L^{(t)}(s)}$$

end for

Hedge on a continuum

Hedge on S with learning rates (η_t).

```
for t ∈ ℕ do
    Play ∼ x^(t)
    Reveal ℓ^(t) : S → [0, M]
    Update
        x^(t+1)(s) ∝ x^(0)(s) e^(−η_{t+1} L^(t)(s))
end for
```

$$x^{(t+1)}(s) \propto x^{(0)}(s)e^{-\eta_{t+1}L^{(t)}(s)}$$

One interpretation: instance of the dual averaging method

$$x^{(t+1)} \in \operatorname*{arg\,min}_{x\in\Delta(S)} \left\langle L^{(t)}, x\right\rangle + \frac{1}{\eta_{t+1}}\psi(x)$$

with

- Hilbert space $\mathcal{H} = L^2(S)$, $\langle \ell, x\rangle = \int_S \ell(s)x(s)\lambda(ds)$
- $\Delta(S) = \{x \in L^2(S) : x \geq 0,\ \|x\|_1 = 1\}$
- $\psi(x) = \int_S x(s)\ln x(s)\lambda(ds)$

Hedge on a continuum

Basic Regret Bound

For all $x \in \Delta(S)$,

$$R^{(T)}(x) \leq \frac{M^2}{2} \sum_{\tau=1}^{t} \eta_{\tau+1} + \frac{\psi(x)}{\eta_{t+1}}$$

But ψ is unbounded on $\Delta(S)$.

Hedge on a continuum

Basic Regret Bound

For all $x \in \Delta(S)$,

$$R^{(T)}(x) \leq \frac{M^2}{2} \sum_{\tau=1}^{t} \eta_{\tau+1} + \frac{\psi(x)}{\eta_{t+1}}$$

But ψ is unbounded on $\Delta(S)$.

Take $x = \frac{1}{\lambda(A)} 1_A$ for some $A \subset S$. Then

$$\psi(x) = \int_S x(s) \ln x(s) \lambda(ds) = \ln \frac{1}{\lambda(A)}$$

can be arbitrarily large for arbitrarily small A.

Working around unbounded regularizers

Idea:

- Call $s_t^\star \in \arg\min_{s \in S} L^{(t)}(s)$ (L supposed continuous).

$$\begin{aligned}
R^{(t)}(x) &= \sum_{\tau=1}^{t} \left\langle \ell^{(\tau)}, x^{(\tau)} - x \right\rangle \\
&\leq \sum_{\tau=1}^{t} \left\langle \ell^{(\tau)}, x^{(\tau)} - \delta_{s_t^\star} \right\rangle \\
&= \sum_{\tau=1}^{t} \left\langle \ell^{(\tau)}, x^{(\tau)} - y \right\rangle + \sum_{\tau=1}^{t} \left\langle \ell^{(\tau)}, y - \delta_{s_t^\star} \right\rangle \\
&= R^{(t)}(y) + \left\langle L^{(t)}, y - \delta_{s_t^\star} \right\rangle
\end{aligned}$$

- Take $y \in \mathcal{B}_t$, set of distributions supported near $s_t^\star$

Revised regret bound

$$\sup_{x \in \Delta(S)} R^{(t)}(x) \leq R^{(t)}(y_0) + \sup_{y \in \mathcal{B}_t} \left\langle L^{(t)}, y - \delta_{s_t^\star} \right\rangle$$

Working around unbounded regularizers

Take $y_0 = \frac{1}{\lambda(A_t)} 1_{A_t}$

$$\sup_{x \in \Delta(S)} R^{(t)}(x) \leq R^{(t)}(y_0) + \sup_{y \in \mathcal{B}_t} \left\langle L^{(t)}, y - \delta_{s_t^\star} \right\rangle$$

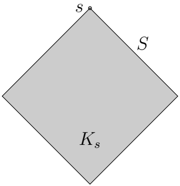
$$R^{(t)}(y_0)$$
$$\leq \frac{M^2}{2} \sum_{\tau=1}^{t} \eta_{\tau+1} + \frac{\psi(y_0)}{\eta_{t+1}}$$
$$\leq \frac{M^2}{2} \sum_{\tau=1}^{t} \eta_{\tau+1} + \frac{1}{\eta_{t+1}} \ln \frac{1}{\lambda(A_t)}$$

$$\left\langle L^{(t)}, y - \delta_{s_t^\star} \right\rangle$$
$$= \int_{A_t} y(s)(L^{(t)}(s) - L^{(t)}(s_t^\star))\lambda(ds)$$
$$\leq Ltd(A_t)$$

Uniformly fat sets

Uniform fatness

S is v-uniformly fat (w.r.t. the measure λ) if
$\forall s \in S$, $\exists$ convex $K_s \subset S$, with $s \in K_s$ and $\lambda(K_s) \geq v$.



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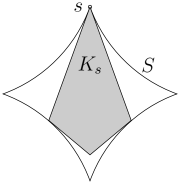

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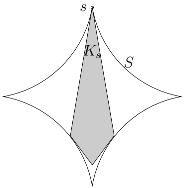

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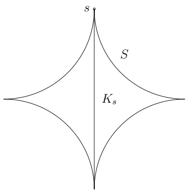


Regret bound on uniformly fat sets

Final bound

$$\sup_{x \in \Delta(S)} R^{(t)}(x) \leq \frac{M^2}{2} \sum_{\tau=1}^{t} \eta_{\tau+1} + \frac{\ln \frac{1}{v}}{\eta_{t+1}} + \frac{n \ln t}{\eta_{t+1}} + Ld(S)$$

Can optimize over η_t to get

$$\sup_{x \in \Delta(S)} R^{(t)}(x) \leq Ld(S) + M\sqrt{t}\sqrt{\frac{n \ln t + \ln \frac{1}{v}}{2}}$$

Beyond Hedge

Dual averaging with learning rates (η_t), strongly convex regularizer ψ

```
for t ∈ ℕ do
    Play x^(t)
    Discover ℓ^(t) ∈ H*
    Update
```

$$x^{(t+1)} = \arg\min_{x \in \Delta(S)} \left\langle L^{(t)}, x \right\rangle + \frac{1}{\eta_{t+1}} \psi(x) \tag{1}$$

```
end for
```

Beyond Hedge

Dual averaging with learning rates (η_t), strongly convex regularizer ψ

for $t \in \mathbb{N}$ **do**
    Play $x^{(t)}$
    Discover $\ell^{(t)} \in \mathcal{H}^*$
    Update

$$x^{(t+1)} = \arg\min_{x \in \Delta(S)} \left\langle L^{(t)}, x \right\rangle + \frac{1}{\eta_{t+1}} \psi(x) \tag{1}$$

end for

- $\mathcal{H}$ is infinite dimensional. Can we solve

$$\min_{x \in \Delta(S)} \left\langle L^{(t)}, x \right\rangle + \frac{1}{\eta_{t+1}} \psi(x)$$

Beyond Hedge

Dual averaging with learning rates (η_t), strongly convex regularizer ψ

for $t \in \mathbb{N}$ **do**
  Play $x^{(t)}$
  Discover $\ell^{(t)} \in \mathcal{H}^*$
  Update

$$x^{(t+1)} = \arg\min_{x \in \Delta(S)} \left\langle L^{(t)}, x \right\rangle + \frac{1}{\eta_{t+1}} \psi(x) \tag{1}$$

end for

- $\mathcal{H}$ is infinite dimensional. Can we solve

$$\min_{x \in \Delta(S)} \left\langle L^{(t)}, x \right\rangle + \frac{1}{\eta_{t+1}} \psi(x)$$

- Can we obtain a sublinear regret bound?

$$\sup_{x \in \Delta(S)} R^{(t)}(x) \leq \frac{M^2}{2} \sum_{\tau=1}^{t} \eta_{\tau+1} + \frac{1}{\eta_{t+1}} \psi(y) + Ltd(A_t)$$

Outline

Numerical Example

Hedge algorithm on hollow cube in $\mathbb{R}^3$.

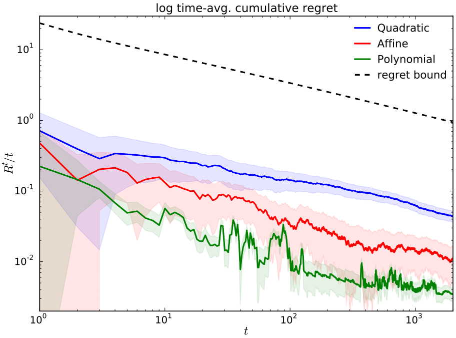
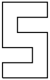


Figure: Per-round regret

Numerical Example

Hedge algorithm

- on set S
- with Lipschitz losses
- with $\eta_t = \Theta\sqrt{\frac{\ln t}{t}}$



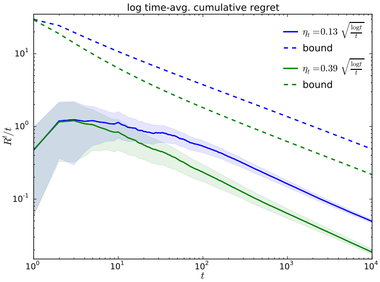


Figure: Effect of the learning rate $\eta_t = \theta\sqrt{\frac{\ln t}{t}}$

Numerical Example

Figure: Evolution of the Hedge density

Conclusion

Summary

- Can learn on a continuum, when losses are Lipschitz and S has reasonable geometry.
- Similar guarantee to learning on a cover, but do not need to maintain a cover.
- Can generalize to the dual averaging method.

Extensions and open questions

- Bandit formulation, e.g. [3].
- Regret lower bound.
- When is it easy to sample from the Hedge distribution?

[3]Sébastien Bubeck, Rémi Munos, Gilles Stoltz, and Csaba Szepesvari. X-armed bandits. *Journal of Machine Learning Research (JMLR)*, 12(12):1587–1627, 2011

Thank you

Thank you.

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[1] David Blackwell. An analog of the minimax theorem for vector payoffs. *Pacific Journal of Mathematics*, 6(1):1–8, 1956.

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[8] Martin Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. In *ICML*, pages 928–936, 2003.

Additional slides: Regret bound on uniformly fat sets

$$\sup_{x\in\Delta(S)} R^{(t)}(x) \leq \frac{M^2}{2}\sum_{\tau=1}^{t}\eta_{\tau+1} + \frac{1}{\eta_{t+1}}\ln\frac{1}{\lambda(A_t)} + Ltd(A_t)$$

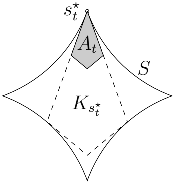

Figure: $A_t = s_t^\star + d_t(K_{s_t^\star} - s_t^\star)$. Then $\lambda(A_t) \geq d_t^n v$ and $d(A_t) \leq d_t d(S)$.

$$\sup_{x\in\Delta(S)} R^{(t)}(x) \leq \frac{M^2}{2}\sum_{\tau=1}^{t}\eta_{\tau+1} + \frac{1}{\eta_{t+1}}\ln\frac{1}{vd_t^n} + Ltd_t d(S)$$

Additional slides: Hedge Vs. learning on a cover

- Given a horizon T and a cover $\mathcal{A}_T$ with $d(A) \leq d_T d(S)$ for all $A \in \mathcal{A}_T$.
- Run discrete Hedge on elements of the cover.

Additional slides: Hedge Vs. learning on a cover

- Given a horizon T and a cover $\mathcal{A}_T$ with $d(A) \leq d_T d(S)$ for all $A \in \mathcal{A}_T$.
- Run discrete Hedge on elements of the cover.
- Then

$$R^{(T)}(x) \leq \underbrace{\frac{M^2 T\eta}{8} + \frac{\ln|\mathcal{A}_T|}{\eta}}_{\text{Discrete Hedge}} + \underbrace{Ld(S)d_T}_{\text{Additional regret}}$$

- With $|\mathcal{A}_T| \approx \frac{1}{d_T^n}$,

$$R^{(T)}(x) \leq \frac{M^2 T\eta}{8} + \frac{\ln \frac{1}{d_T^n}}{\eta} + LD(S)d_T$$

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$$R^{(T)}(x) \leq \frac{M^2 T\eta}{8} + \frac{\ln \frac{1}{d_T^n}}{\eta} + LD(S)d_T$$

- Have to explicitly compute a (hierarchical) cover.