Online learning on a continuum

Online decision problem on $S$.

1. The decision maker chooses distribution $x(t)$ over $S$.
2. A loss function $L(x(t)) : S \to [0, M]$ is revealed. Assumed $L$-Lipschitz.
3. The decision maker incurs expected loss
   \[ E(x(t), \ell(t)) = \int_S x(t)(s) \ell(s) \mathrm{d}s = E_x[\ell(t)] \]

### Regret

- **Objective**: design algorithm with
  \[ \sup_{x(t) \in \Delta^T} \sup_{x(t)} R^T(x) = o(T) \]

- **Regret rates**

<table>
<thead>
<tr>
<th>Assumptions on $x(t)$</th>
<th>convex</th>
<th>$\alpha$-exp-concave</th>
<th>uniformly $L$-Lipschitz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assumptions on $S$</td>
<td>convex</td>
<td>convex</td>
<td>$\nu$-uniformly fat</td>
</tr>
<tr>
<td>Method</td>
<td>Gradient (Zinkevich)</td>
<td>Hedge (Hazan et al.)</td>
<td>Hedge (this work)</td>
</tr>
<tr>
<td>Learning rates</td>
<td>$1/\sqrt{T}$</td>
<td>$\alpha$</td>
<td>$1/\sqrt{T}$</td>
</tr>
<tr>
<td>$R^T(x)$</td>
<td>$O(\sqrt{T})$</td>
<td>$O((\log t)$</td>
<td>$O(\sqrt{\log t})$</td>
</tr>
</tbody>
</table>

### Table: Regret upper bounds for different classes of losses.

#### Basic Regret Bound

- For all $x \in \Delta(S)$,
  \[ R^T(x) \leq M^2 \sum_{t=1}^T \eta_t + \frac{\psi(x)}{\eta_t} \]

  - In the finite case, $\psi(x) \leq \ln N$ on $\Delta^N$.
  - But $\psi$ is unbounded on $\Delta(S)$.
  - Take $x = \frac{1}{|A|} s$ for some $A \subset S$. Then $\psi(x) = \int_A x(s) \log x(s) \mathrm{d}s = \ln \frac{|S|}{|A|}$

#### Working around unbounded regularizers

For any density $y$,
\[ R^T(x) \leq R^T(y) + \int_{\Delta(S)} y - \psi(x) \mathrm{d}s \]

In particular, if $y$ is uniform on $A_t$ which contains $s^* = \arg \min_{s \in \Delta(S)} L(x(s))$,
\[ \sup_{x \in \Delta(S)} R^T(x) \leq R^T(y) + \int_{\Delta(S)} y - \psi(x) \mathrm{d}s \]

\[ \leq \frac{M^2}{2} \sum_{t=1}^T \eta_t + \frac{1}{\eta_t} \ln \frac{1}{\lambda(A_t)} + L d(A_t) \]

#### Uniformly fat sets

- $S$ is $\nu$-uniformly fat (w.r.t. the measure $\lambda$) if $\forall s \in S$, $\exists$ convex $K_s \subset S$, with $s \in K_s$ and $\lambda(K_s) \geq \nu$.

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#### Regret bound on uniformly fat sets

Take $A_t = s^* + d_t(K_s) - s^*$. Then
\[ \lambda(A_t) \geq d_t \nu \]
\[ d_t(A_t) \leq d_t(S) \]

**Final bound**

\[ \sup_{s \in \Delta(S)} R^T(x) \leq L d(S) + M \sqrt{\frac{n \ln t + \nu}{2}} \]

### Generalizing to dual averaging

**Dual averaging with learning rates $\eta_t$**, strongly convex regularizer $\psi$.

1. For $t \in \mathbb{N}$ do
2. Play $x(t)$
3. Discover $\ell(t) \in \mathcal{H}^*$
4. Update $x(t+1) = \arg \min_{x \in \Delta(S)} \langle \ell(t), x \rangle + \frac{1}{\eta_t} \psi(x)$

- $\mathcal{H}$ is infinite dimensional. Can we solve $\min_{x \in \Delta(S)} \langle \ell(t), x \rangle + \frac{1}{\eta_t} \psi(x)$?
- Can we obtain a sublinear regret bound?

Yes, for a family of strongly convex $f$-divergences.

### Numerical experiments

- **Hedge algorithm**
  - on set $S$
  - with Lipschitz losses
  - with $\eta_t = \frac{1}{\sqrt{t}}$

- **Effect of the learning rate $\eta_t = \frac{1}{\sqrt{t}}$**

- **Examples of uniformly fat sets (left) and a not uniformly fat set (right)**

### Learning on a cover

- Given a horizon $T$ and a cover $\mathcal{A}_T$ with $d(A) \leq d_f(S)$ for all $A \in \mathcal{A}_T$.
- Run discrete Hedge on elements of the cover.

Then
\[ R^T(x) \leq M^2 T \eta_t + \frac{\ln |\mathcal{A}_T|}{\eta_t} + L d(S) d_T 
\]

**Additional regret**

- With $|\mathcal{A}_T| \approx T$, $R^T(x) \leq M^2 T \eta_t + \frac{\ln T}{\eta_t} + L d(S) d_T$
- Have to explicitly compute a (hierarchical) cover.

### Conclusion

**Summary**

- Can learn on a continuum, when losses are Lipschitz and $S$ has reasonable geometry.
- Similar guarantee to learning on a cover, but do not need to maintain a cover.
- Can generalize to the dual averaging method.
- Extensions and open questions
- Bandit formulation
- Regret lower bound
- When is it easy to sample from the Hedge distribution?