Stability of Nash equilibria in Congestion Games under Replicator Dynamics

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Outline

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Introduction

Class of Congestion Games, under Replicator Dynamics.

Congestion games

- Population of players (non atomic), with action set $\mathcal{A}$
- Mass distribution $x \in \Delta^{\mathcal{A}}$ determines losses $\ell(x) \in \mathbb{R}_+^{\mathcal{A}}$

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Replicator dynamics

Players' mass distribution obeys ODE

$$\dot{x}_a(t) = x_a(t) \left(\langle \ell(x(t)), x(t) \rangle - \ell_a(x(t)) \right)$$
$$x_a(0) \text{ given}$$

Our goal: study stability of equilibria.

Example: routing game

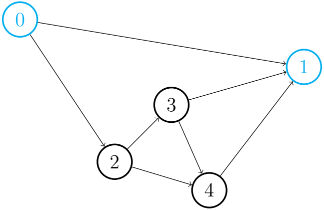

Figure: Routing game

- Population: packet routers / drivers.
- Action set $\mathcal{A}$: paths from (0) to (1)
- Mass distribution x determines, edge loads Mx, edge costs $c(Mx)$.
- Loss function:

$$\ell_a(x) = M_a^T c(Mx)$$

$M \in \mathbb{R}^{\mathcal{E}\times\mathcal{A}}$: path-edge incidence matrix of the graph.

Nash equilibria

Nash equilibria $\mathcal{N}$

A mass distribution x^* is a Nash equilibrium if $\forall x \in \Delta^{\mathcal{A}}$,

$$\langle \ell(x^*), x^* - x \rangle \leq 0$$

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Replicator dynamics

Replicator dynamics

For $a \in \mathcal{A}$

$$\dot{x}_a(t) = x_a(t)\left(\langle \ell(x(t)), x(t)\rangle - \ell_a(x(t))\right)$$
$$x_a(0) \text{ given}$$

Model:

- Randomly match players
- Compare actions a and a'
- Player with higher loss replicates action of other player with probability $\ell_a - \ell_{a'}$.

Stationary points

Stationary points $\mathcal{R}$

$$\dot{x} = 0 \Leftrightarrow \forall a,\ x_a \left(\langle \ell(x), x\rangle - \ell_a(x)\right) = 0$$
$$\Leftrightarrow \ell_a(x) \text{ constant on the support of } x$$

Nash equilibria $\mathcal{N}$

$$x \in \mathcal{N} \Leftrightarrow \langle \ell(x), x\rangle \leq \langle \ell(x), y\rangle \ \ \forall y$$
$$\Leftrightarrow \ell_a(x) \text{ constant and minimal on the support of } x$$

Theorem [1]

Solution trajectories converge to $\mathcal{R}$.

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[1] Simon Fischer and Berthold Vöcking. On the evolution of selfish routing. In *Algorithms–ESA 2004*, pages 323–334. Springer, 2004

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- Stability of equilibria?

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Stability of replicator dynamics

$$\dot{x}_a(t) = F_a(x(t)) = x_a(t)\left(\langle \ell(x(t)), x(t)\rangle - \ell_a(x(t))\right)$$

Instability of non-Nash equilibria

If $x \in \mathcal{R} \setminus \mathcal{N}$, then x is unstable.

proof:

- $\mathcal{H} = \sum_{a\in\mathcal{A}} x_a = 0$, then $F(\Delta) \subset \mathcal{H}$.
- Derive Jacobian $\tilde{\nabla} F$ of F restricted to $\mathcal{H}$.
- If $\mathcal{A}^*$ is the support of x and $\mathcal{A}^\diamond$ its complement, then

$$Sp(\tilde{\nabla} F(x)) \supset \{\langle \ell(x), x\rangle - \ell_a(x)\}_{a\in\mathcal{A}^\diamond}$$

Stability of replicator dynamics

Exponential stability

If M is injective, then

$$x \in \mathcal{N} \Leftrightarrow x \text{ is locally exponentially stable}$$

proof:
If M is injective, can show that $\tilde{\nabla} F(x)$ is negative definite.

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Simulation

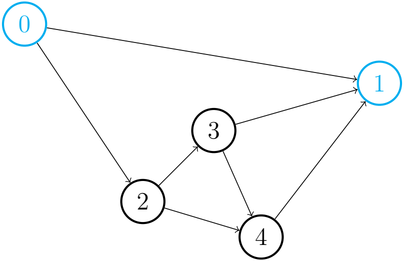

Figure: Routing game with $|\mathcal{A}| = 4$.

Simulation

$$\mathcal{N} = \{x : x_1 = .757, x_2 = 0, x_3 + x_4 = .2426\}$$

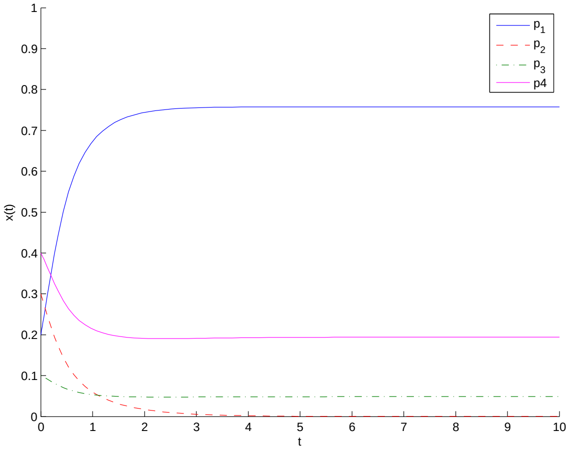

Figure: Masse trajectories $x_a(t)$, $a \in \{p_1, p_2, p_3, p_4\}$

Simulation

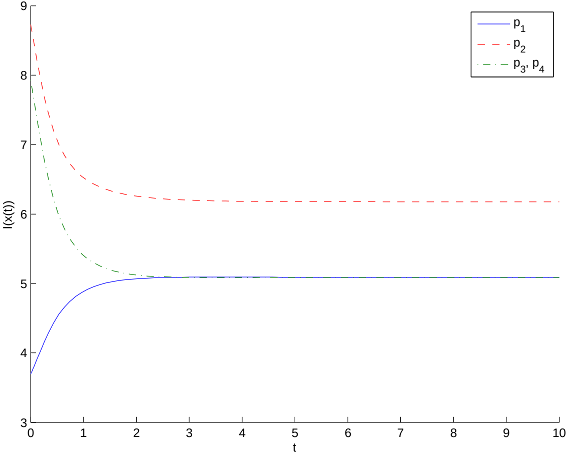

Figure: Losses $\ell_a(x(t))$

Simulation

$$\dot{x}_a(t) = x_a(t)\left(\langle \ell(x(t)), x(t)\rangle - \ell_a(x(t))\right)$$

$\mathcal{N}$: Nash equilibria
$\mathcal{R}$: Stationary points

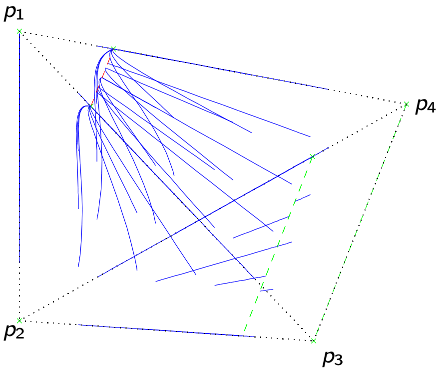

Figure: Mass trajectories in the simplex: convergence to restricted equilibria

Outline

Extension to discrete time

Convergence of discrete time dynamics: approximate replicator dynamics [2]

$$x_a^{(t+1)} - x_a^{(t)} = \eta_t x_a^{(t)} \left(\left\langle \ell(x^{(t)}), x^{(t)} \right\rangle - \ell_a(x^{(t)}) \right) + \eta_t U_a^{(t)}$$

- $U_a^{(t)}$: stochastic perturbation term.
- η_t: discretization time steps, $\sum_t \eta_t = \infty$.

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[2] Walid Krichene, Benjamin Drighès, and Alexandre Bayen. On the convergence of no-regret learning in selfish routing.
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Extension to discrete time

Convergence of discrete time dynamics: approximate replicator dynamics [2]

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- $U_a^{(t)}$: stochastic perturbation term.
- η_t: discretization time steps, $\sum_t \eta_t = \infty$.

Convergence of approximate replicator dynamics

$x^{(t)} \to \mathcal{N}$ almost surely, under mild conditions on $U^{(t)}, \eta_t$.

e.g. $\sup_t \mathbb{E} \|U^{(t)}\|^q < \infty$ and $\sum_t \eta_t^{1+\frac{q}{2}} < \infty$

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Congestion games, under replicator dynamics

- Trajectories converge to stationary points $\mathcal{R}$
- Non-Nash equilibria $\mathcal{R} \setminus \mathcal{N}$ are unstable
- If injective incidence matrix: $\mathcal{N} \Leftrightarrow$ loc. exp. stable equilibrium
- Extension to discrete-time dynamics

Thank you!

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[1] Simon Fischer and Berthold Vöcking. On the evolution of selfish routing. In *Algorithms–ESA 2004*, pages 323–334. Springer, 2004.

[2] Walid Krichene, Benjamin Drighès, and Alexandre Bayen. On the convergence of no-regret learning in selfish routing. In *31st International Conference on Machine Learning (ICML)*. JMLR, 2014.

[3] Walid Krichene, Syrine Krichene, and Alexandre Bayen. Convergence of mirror descent dynamics. In *European Control Conference (ECC), in review*, 2015.

[4] Robert W Rosenthal. A class of games possessing pure-strategy nash equilibria. *International Journal of Game Theory*, 2(1):65–67, 1973.