Stackelberg Thresholds on Parallel Networks with Horizontal Queues

Yasser Jebari Walid Krichene Jack Reilly Alexandre Bayen

Electrical Engineering and Computer Sciences, UC Berkeley
Ecole Polytechnique, France

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Outline

1. Routing Games and Stackelberg Routing
    - Routing Game
    - Stackelberg routing game
    - The Non-compliant First (NCF) strategy
2. Stackelberg thresholds
    - Definition
    - Main results
    - Numerical example

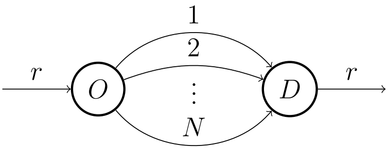
Outline

Congestion Routing Game

- Parallel network, N edges (links)
- Continuum of players $[0, r]$. Total flow demand r (jobs/s, cars/s)
- Action set of player $p \in [0, r]$: routes $\{1, \ldots, N\}$

$$A : [0, r] \to \{1, \ldots, N\}$$

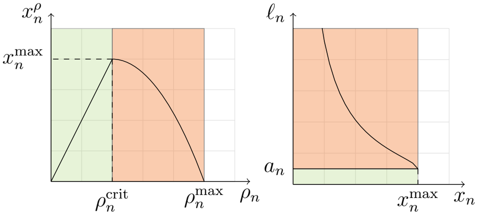
- Cost for any player p with $A(p) = n$: latency $\ell_n(x_n, m_n)$. Depends on
  - total flow x_n on link n. $x_n = m(A^{-1}(\{n\}))$. $x_n \in [0, x_n^{\max}]$
  - congestion state $m_n \in \{0, 1\}$
- Game entirely described by vectors x and m
- Total cost: $C(x, m) = \sum_n x_n \ell_n(x_n, m_n)$



Model assumptions

$$\ell_n : [0, x_n^{\max}] \times \{0, 1\} \to \mathbb{R}_+$$

- $\ell_n(\cdot, 0) = a_n$ constant
- $\ell_n(\cdot, 1)$ is decreasing from $(0, x_n^{\max})$ onto $(a_n, +\infty)$
- $\lim_{x_n \to x_n^{\max}} \ell_n(x_n, 1) = a_n$



Assume $a_1 < a_2 < \cdots < a_N$

Nash equilibria

Nash equilibrium

(x, m) is a Nash equilibrium if

$$n \in \text{supp}(x) \Rightarrow \forall k,\ \ell_n(x_n, m_n) \le \ell_k(x_k, m_k)$$

Example:

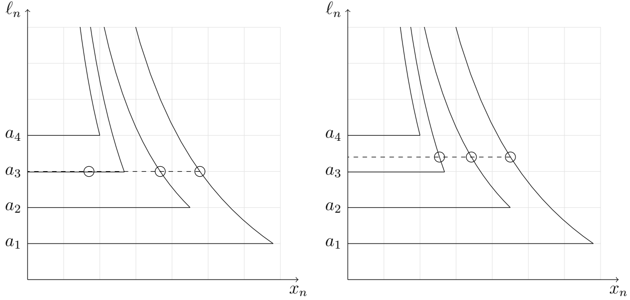


Figure : Single-link-free-flow equilibrium (left) and congested equilibrium (right)

Nash equilibria

- No essential uniqueness.
- But at most $2N$ equilibria

Best Nash equilibrium (one with minimal cost)

- is a single-link free-flow eq
- has minimal support
- can be computed in $O(N^2)$ time

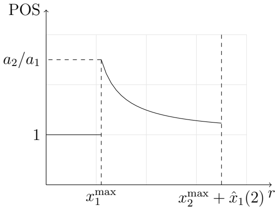


Figure : Price of stability of a 2-link network

Outline

Stackelberg routing game

Stackelberg Game: A central authority (leader) has control over a fraction α of the total flow.
Example: through a phone app.

- some drivers are altruistic
- incentivize drivers to participate, e.g. through lotteries

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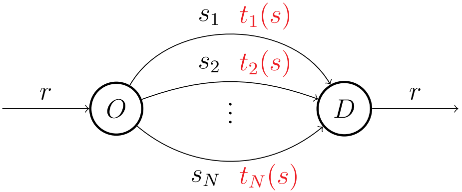
Stackelberg game

- Leader: route "compliant" players (αr): strategy $s \in \prod_{n=1}^{N} [0, x_n^{\max}]$
- Followers ($(1-\alpha)r$) choose their routes selfishly after s is revealed strategy $t(s), m(s)$



Leader seeks to minimize total cost $C(s + t(s), m(s))$

How can Stackelberg routing help?

Simple example

- $r = 10$ units of flow.
- control $\alpha = .04$

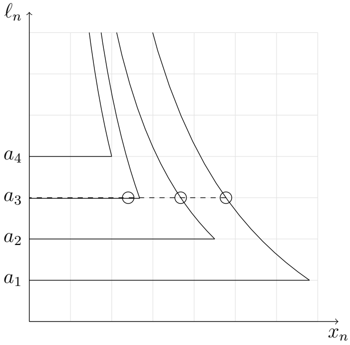


Figure : Best Nash equilibrium with $\alpha = 0$ (left) and $\alpha = .04$ (right)

Note: this strategy s is not optimal.

Optimal Stackelberg Strategy

Definition: Optimal Stackelberg strategy

s^* is an optimal strategy if the cost of the induced equilibrium

$$C(s^* + t(s^*), m(s^*))$$

is minimal over $S(N, r, \alpha)$

Outline

Non-compliant First

Non-compliant first strategy $\bar{s}$

- Compute BNE of non-compliant flow alone: $(\bar{t}, \bar{m}) = \text{BNE}(N, (1-\alpha)r)$
- Compute $\bar{s}$: Assign the compliant flow by filling remaining links (non congested by non-compliant) each to max capacity.

Can compute it in P time

Theorem

The NCF strategy $\bar{s}$ is optimal.

Non-compliant first (NCF)

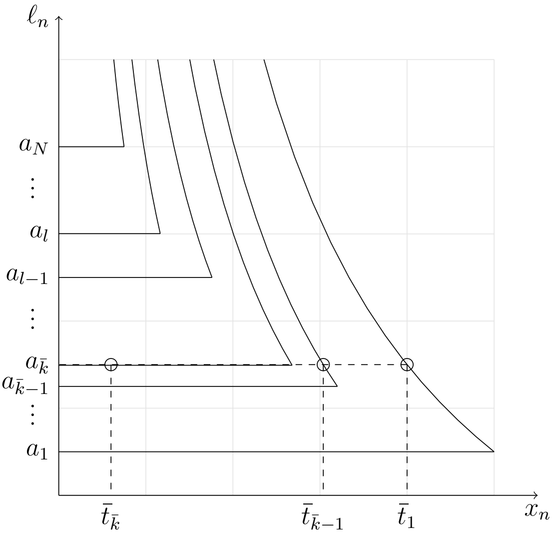
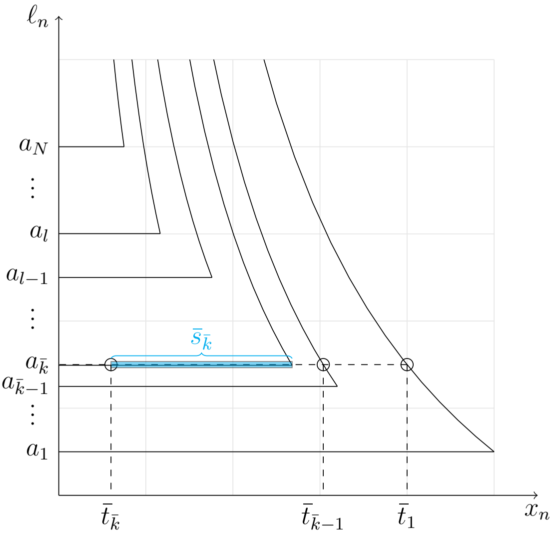


Figure : Non-compliant first strategy $\bar{s}$

Non-compliant first (NCF)



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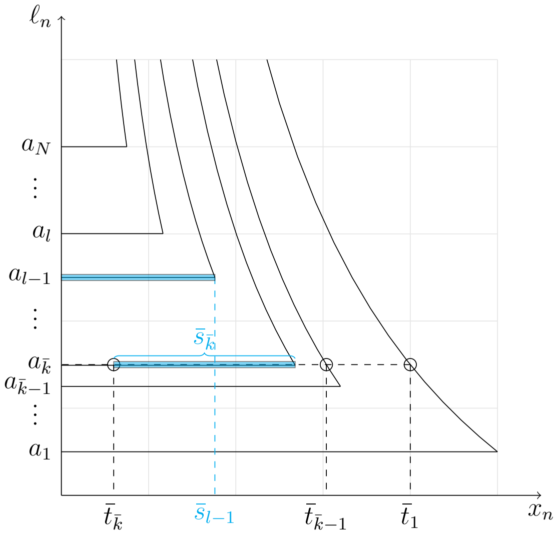

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Non-compliant first (NCF)

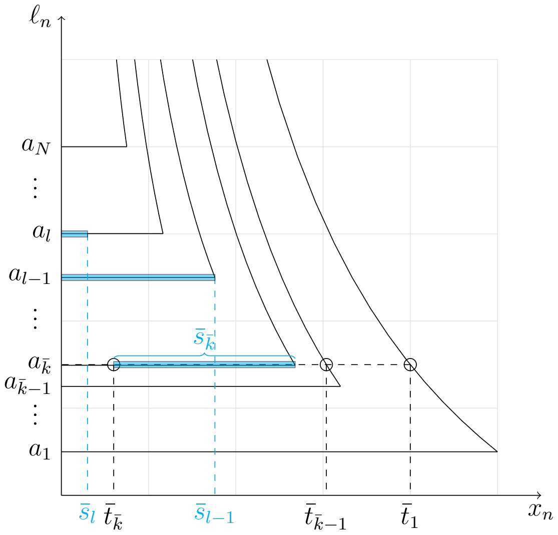


Figure : Non-compliant first strategy $\bar{s}$

Set of optimal Stackelberg strategies

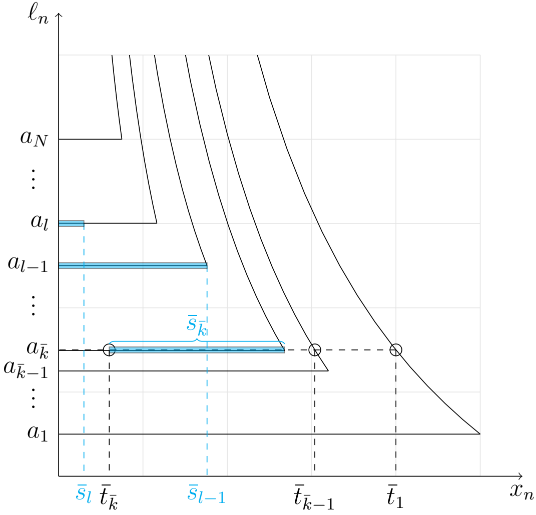
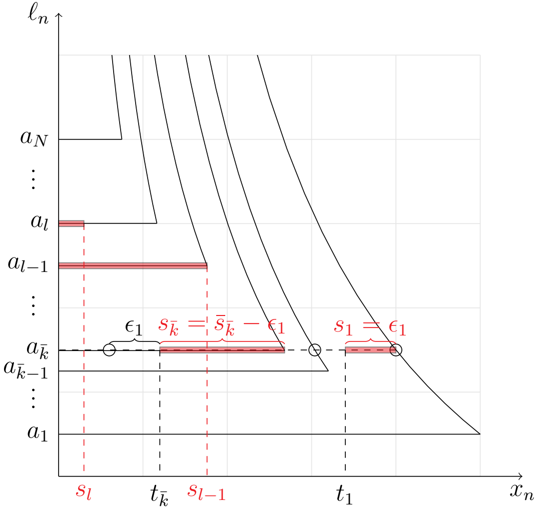


Figure : Example of an optimal Stackelberg strategy $s = \bar{s} - \epsilon$.

Set of optimal Stackelberg strategies



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Outline

Stackelberg thresholds

Definition

Given a demand r, the Stackelberg threshold is the smallest α such that the induced Stackelberg eq. with compliance rate α has strictly lower cost

$$\alpha(r) = \inf\{\alpha : C_{\mathsf{NCF}}(\alpha) < C_{\mathsf{Nash}}\}$$

Outline

Main results

- For fixed r, cost function is piecewise-constant in α. Discontinuities are a subset of
$$\left\{ \frac{r - r^{\text{NE}}(k)}{r}, k \in \{1, \dots, N\} \right\}$$
- $r^{\text{NE}}(k)$ maximum demand for which Nash eq. has support $\{1, \dots, k\}$

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- For fixed r, cost function is piecewise-constant in α. Discontinuities are a subset of
$$\left\{ \frac{r - r^{\mathrm{NE}}(k)}{r}, k \in \{1, \dots, N\} \right\}$$
- $r^{\mathrm{NE}}(k)$ maximum demand for which Nash eq. has support $\{1, \dots, k\}$
- Explicit expression for the cost
- Explicit expression for the Stackelberg threshold as a function of r

$$\alpha(r) = 1 - \frac{r^{\mathrm{NE}}(k(r))}{r}$$

Stackelberg thresholds illustration

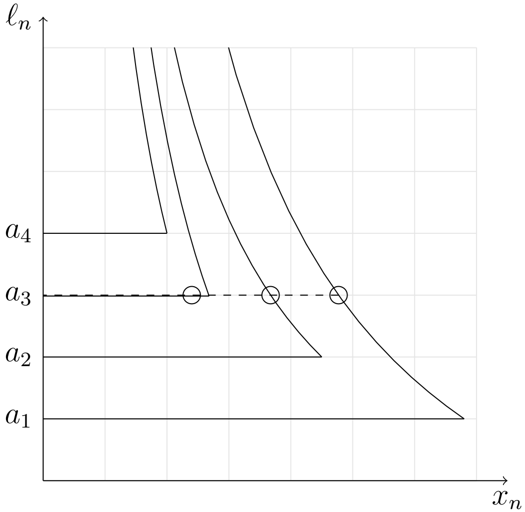
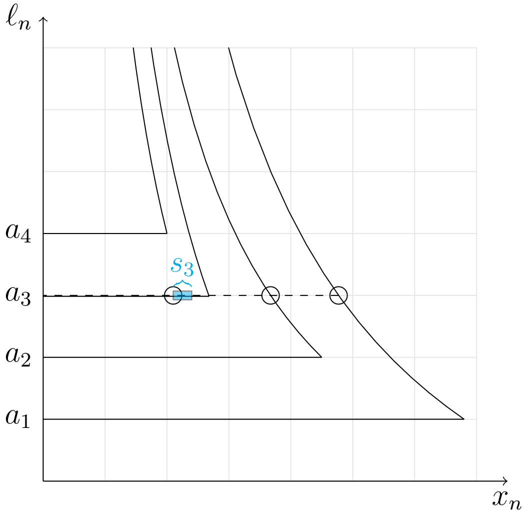


Figure : Illustration on a simple example

Stackelberg thresholds illustration



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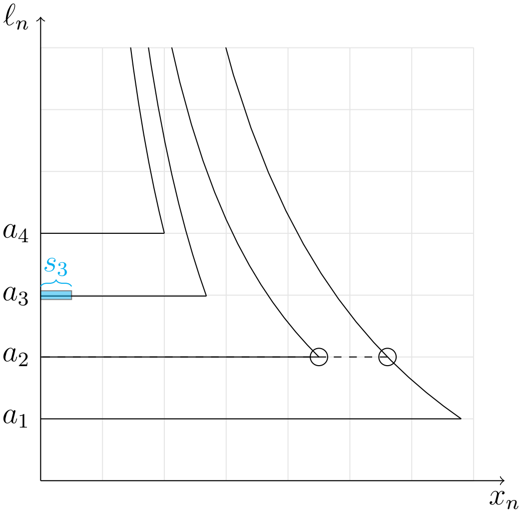

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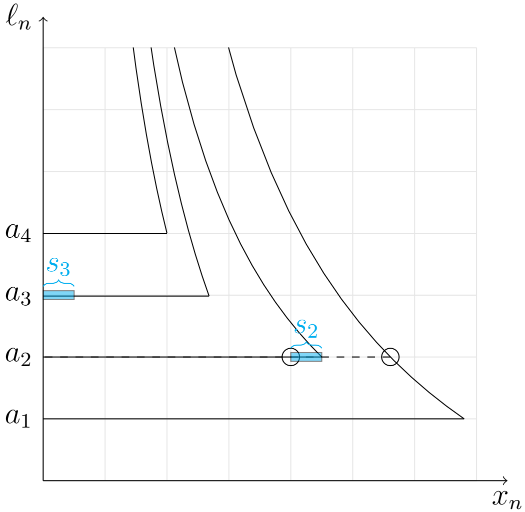

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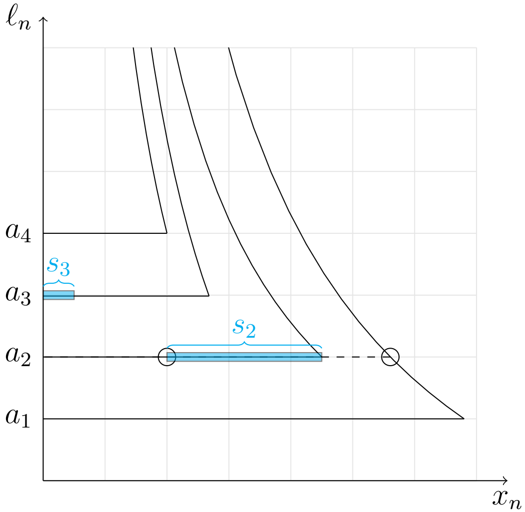

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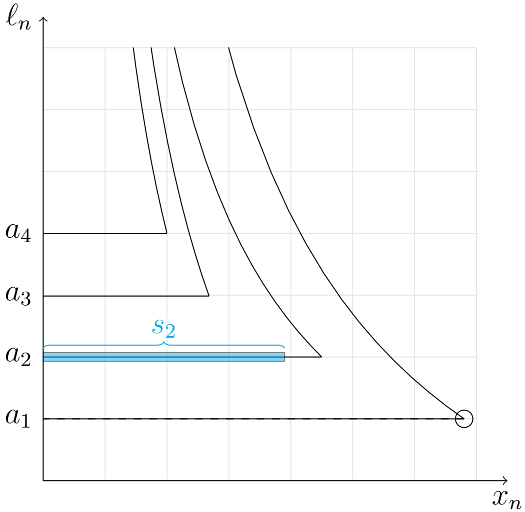

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Outline

Numerical example

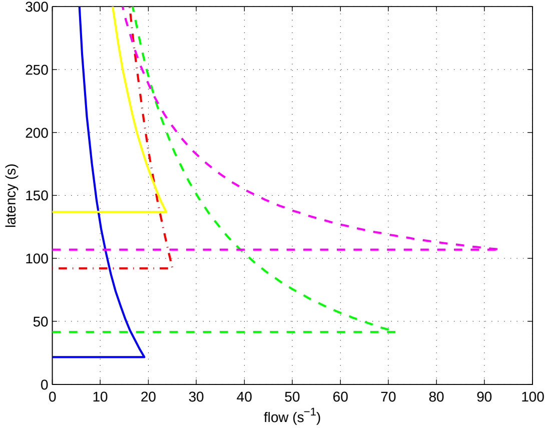

Figure : Latency functions

Numerical example

- For fixed r, cost function is piecewise-constant in α. Discontinuities are a subset of

$$\left\{ \frac{r - r^{\text{NE}}(k)}{r}, k \in \{1, \dots, N\} \right\}$$

- $r^{\text{NE}}(k)$ maximum demand for which Nash eq. has support $\{1, \dots, k\}$

Numerical example

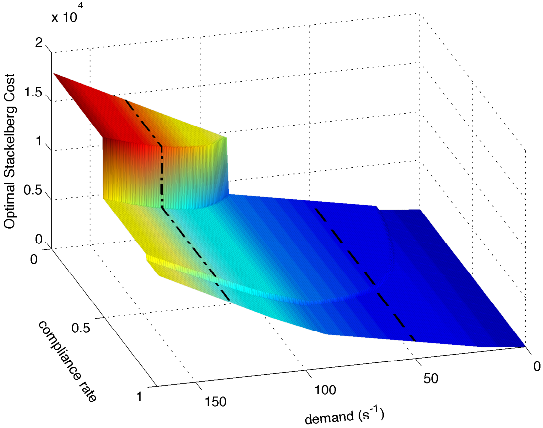

Figure : Cost as a function of demand r and compliance rate α

Numerical example

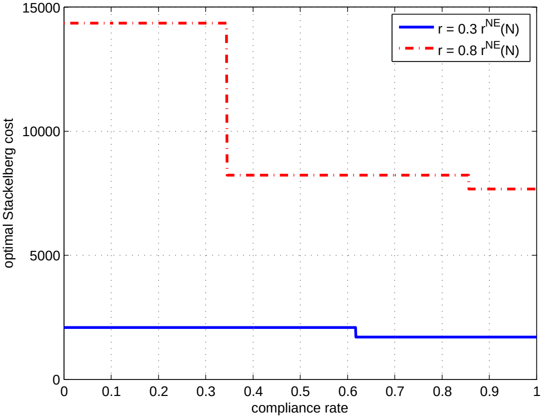

Figure : Cost as a function of compliance rate α for fixed demand

Numerical example

- Explicit expression for the Stackelberg threshold as a function of r

$$\alpha(r) = 1 - \frac{r^{\mathrm{NE}}(k(r))}{r}$$

Numerical example

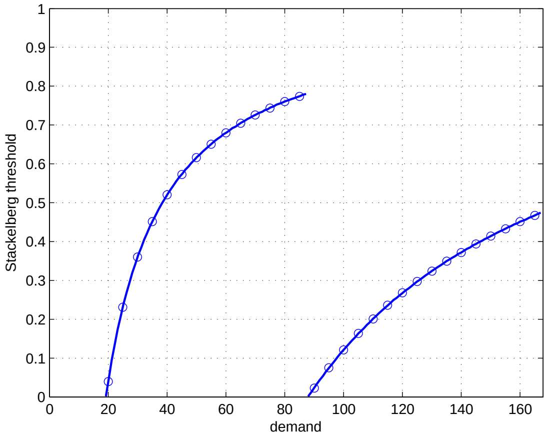

Figure : Latency functions

Summary and future work

- Class of latency functions designed for physical networks (transportation).
- In the parallel network case, can solve for Nash equilibria and optimal Stackelberg strategies in $O(N^2)$ time.
- Explicit expression for the Stackelberg thresholds and the optimal cost function

Summary and future work

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- Explicit expression for the Stackelberg thresholds and the optimal cost function

Possible extensions

- More general network topologies
- Heterogeneous case: more than one player type, with different cost functions.
- Repeated routing, limited information, online learning.

[1] Y. Jebbari, W. Krichene, J. Reilly, A. Bayen. "Stackelberg Thresholds on Parallel Networks with Horizontal Queues", 52nd IEEE Conference on Decision and Control.

[2] W. Krichene, J. Reilly, S. Amin, A. Bayen. "Stackelberg Routing on Parallel Networks with Horizontal Queues", 51st IEEE Conference on Decision and Control.

[3] W. Krichene, J. Reilly, S. Amin, A. Bayen. "Characterization and Computation of Nash Equilibria on Parallel Networks with Horizontal Queues", 51st IEEE Conference on Decision and Control.

Thank you.