A Heterogeneous Routing Game

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Outline

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1. The non-atomic routing game
2. An equivalent finite-player game
3. Heterogeneous routing games and potential games
4. Tolling

Introduction

- Most studies of routing games assume that drivers or vehicles are of the same type.
- Motivated by
  - Transportation networks: drivers only care about the travel time (Wardrop, 1952);
  - Packet routing in communication networks (Altman et al., 2006; Banner & Orda, 2007; Czumaj, 2004).

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  - Fuel consumption (Farokhi & Johansson, 2013; Alam et al., 2010);
  - Sensitivity to Latency (Stern, 1999; Stern & Richardson, 2005);
  - Sensitivity to Tolls (Inregia, 2001; Engelson & Lindberg, 2006).

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- Heterogeneous routing games have been considered (Baldacci et al., 2008; Engevall et al., 2004; Fleischer et al., 2004; Fotakis et al., 2010; Karakostas & Kolliopoulos, 2004; Marcotte & Zhu, 2009).
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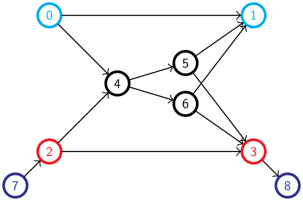
- However, they adjust the sensitivity of the agents either to the observed latencies or the tolls through a multiplicative weight.
- More general classes of latency functions?

Heterogeneous Routing Game



- A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ models the transportation network.
- A set of commodities $\{(s_k, t_k)\}_{k=0}^{K}$.

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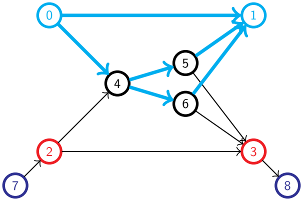

- A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ models the transportation network.
- A set of commodities $\{(s_k, t_k)\}_{k=0}^{K}$.
- $\mathcal{P}_k$: set of all admissible paths over the graph $\mathcal{G}$ that connect $s_k \in \mathcal{V}$ to $t_k \in \mathcal{V}$.
- Denote $\mathcal{P} = \cup_{k=1}^{K} \mathcal{P}_k$.

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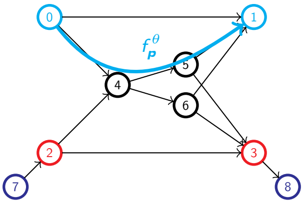

- The type of a player is determined by $\theta \in \Theta$ where Θ is a finite set.
- $f_p^\theta \in \mathbb{R}_{\geq 0}$ denotes the flow of players of type $\theta \in \Theta$ that use path $p \in \mathcal{P}$.

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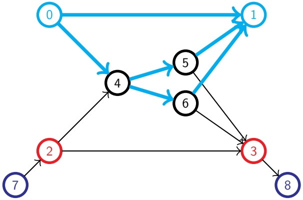

$$f_{p_1}^{\theta} + f_{p_2}^{\theta} + f_{p_3}^{\theta} = F_{\theta}^{1}$$

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- $f_p^\theta \in \mathbb{R}_{\geq 0}$ denotes the flow of players of type $\theta \in \Theta$ that use path $p \in \mathcal{P}$.
- Commodity $k \in [\![K]\!] = \{1, \ldots, K\}$ transfers a flow equal to $(F_k^\theta)_{\theta \in \Theta} \in \mathbb{R}_{\geq 0}^{|\Theta|}$.

Feasibility

A flow vector $f = (f_p^\theta)_{p \in \mathcal{P}, \theta \in \Theta} \in \mathbb{R}^{|\mathcal{P}| \cdot |\Theta|}$ is feasible if $\sum_{p \in \mathcal{P}_k} f_p^\theta = F_k^\theta$ for all $k \in [\![K]\!]$ and $\theta \in \Theta$.

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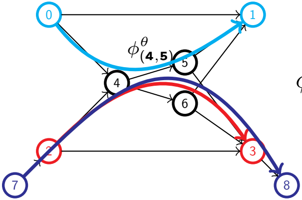

$$\phi^{\theta}_{(4,5)} = f^{\theta}_{p_1} + f^{\theta}_{p'_1} + f^{\theta}_{p''_1}$$

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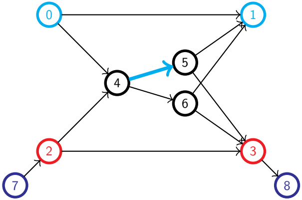
- $\phi_e^{\theta} = \sum_{p \in \mathcal{P}: e \in p} f_p^{\theta}$ denotes the flow of drivers of type θ on edge $e \in \mathcal{E}$.

Heterogeneous Routing Game



$\tilde{\ell}_e^{\theta}((\phi_{(4,5)}^{\theta'})_{\theta' \in \Theta})$

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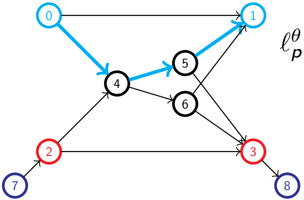

$$\ell_p^\theta(f) = \tilde{\ell}_e^\theta((\phi_{(0,4)}^{\theta'})_{\theta'\in\Theta}) + \tilde{\ell}_e^\theta((\phi_{(4,5)}^{\theta'})_{\theta'\in\Theta}) + \tilde{\ell}_e^\theta((\phi_{(5,1)}^{\theta'})_{\theta'\in\Theta})$$

- Driver of type $\theta \in \Theta$ traveling along $e \in \mathcal{E}$: experiences edge latency $\tilde{\ell}_e^\theta((\phi_e^{\theta'})_{\theta'\in\Theta})$.
- Total latency on path $p \in \mathcal{P}_k$: $\ell_p^\theta(f) = \sum_{e\in p} \tilde{\ell}_e^\theta((\phi_e^{\theta'})_{\theta'\in\Theta})$.

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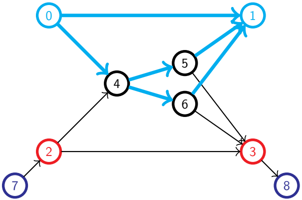

- Driver of type $\theta \in \Theta$ traveling along $e \in \mathcal{E}$: experiences edge latency $\tilde{\ell}_e^{\theta}((\phi_e^{\theta'})_{\theta' \in \Theta})$.
- Total latency on path $p \in \mathcal{P}_k$: $\ell_p^{\theta}(f) = \sum_{e \in p} \tilde{\ell}_e^{\theta}((\phi_e^{\theta'})_{\theta' \in \Theta})$.
- A driver $\Leftrightarrow$ infinitesimal amount of flow, strategically tries to minimize its own latency

$$\min_{p \in \mathcal{P}_k} \ell_p^{\theta}(f)$$

Nash Equilibrium in Heterogeneous Routing Game

Nash Equilibrium*

A flow vector $f = (f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}$ is a Nash equilibrium if for all $k \in [\![K]\!]$ and $\theta \in \Theta$, $f_p^\theta > 0$ for a path $p \in \mathcal{P}_k$ implies that $\ell_p^\theta(f) \leq \ell_{p'}^\theta(f)$ for all $p' \in \mathcal{P}_k$.

*Also called Wardrop equilibrium due to pioneering work of Wardrop (1952), and the fact that pure strategy Nash equilibrium was primarily defined in the context of games with finitely many players. See (Chau & Sim, 2003; Haurie & Marcotte, 1985; Roughgarden & Tardos, 2002).

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For a commodity $k \in [\![K]\!]$ and type $\theta \in \Theta$

- paths with nonzero flow for drivers of type θ have equal costs
- the rest have larger than or equal costs.

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Illustrative Example: Platooning Incentives

- Let $\Theta = \{\mathrm{c}, \mathrm{t}\}$ where t denotes trucks and c denotes cars.
- Edge cost functions

$$\tilde{\ell}_e^{\mathrm{c}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}) = \xi_e(\phi_e^{\mathrm{c}} + \phi_e^{\mathrm{t}}),$$
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- $\gamma_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$: inverse of fuel efficiency of the trucks, function of the flow of trucks: platooning reduces air drag

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Standing Assumptions

Assumption

For all $\theta \in \Theta$ and $e \in \mathcal{E}$, $\tilde{\ell}_e^\theta$ satisfies:

(*i*) $\tilde{\ell}_e^\theta \in \mathcal{C}^1$;

(*ii*) $\tilde{\ell}_e^\theta$ is positive;

(*iii*) $\int_0^{\phi_e^\theta} \tilde{\ell}_e^\theta(u, (\phi_e^{\theta'})_{\theta' \in \Theta \setminus \{\theta\}}) \mathrm{d}u$ is a convex function in ϕ_e^θ.

Assumption (*iii*) can be replaced with the assumption that $\tilde{\ell}_e^\theta((\phi_e^{\theta'})_{\theta' \in \Theta})$ is an non-decreasing function in ϕ_e^θ.

Outline

An equivalent finite-player game

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$$\mathcal{A}_i = \left\{ (f_{p'}^{\theta_i})_{p' \in \mathcal{P}} \in \mathbb{R}_{\geq 0}^{|\mathcal{P}|} \,\middle|\, \sum_{p' \in \mathcal{P}_k} f_{p'}^{\theta_i} = F_k^{\theta_i} \right\}.$$

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- Utility of player i:

$$U_i(a_i, a_{-i}) = \sum_{e \in \mathcal{E}} \int_0^{\phi_e^{\theta_i}} \tilde{\ell}_e^{\theta_i}(u, (\phi_e^{\theta_j})_{\theta_j \in \Theta \setminus \{\theta_i\}}) \mathrm{d}u,$$

- An action profile $a \in \times_{j=1}^{N} \mathcal{A}_j$ is a pure strategy Nash equilibrium if for all $i \in [\![N]\!]$,

$$U_i(a_i, a_{-i}) \geq U_i(\bar{a}_i, a_{-i}), \; \forall \bar{a}_i \in \mathcal{A}_i.$$

An equivalent finite-player game

Lemma

A flow vector $(f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta}$ is a Nash equilibrium of the heterogeneous routing game if and only if $((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},\dots,(f_{p'}^{\theta_N})_{p'\in\mathcal{P}})$ is a pure strategy Nash equilibrium of the abstract game.

proof: write KKT conditions for the problem $\min_{a_i} U_i(a_i, a_{-i})$.

Existence of a Nash Equilibrium

Theorem

The heterogeneous routing game admits at least one Nash equilibrium.

Proof.

Equivalent to showing that the game with $|\Theta|$ players admits a pure strategy Nash equilibrium. The rest is an application of seminal results of (Arrow & Debreu, 1954; Debreu, 1952). □

Outline

Potential Game

The abstract game is a potential game and admits a potential function $V : \times_{i=1}^{N} \mathcal{A}_i \rightarrow \mathbb{R}$ if for all $i \in [\![N]\!]$,

$$V(a_i, a_{-i}) - V(\bar{a}_i, a_{-i}) = U_i(a_i, a_{-i}) - U_i(\bar{a}_i, a_{-i}),$$
$$\forall a_i, \bar{a}_i \in \mathcal{A}_i \text{ and } a_{-i} \in \times_{j \in [\![N]\!] \setminus \{i\}} \mathcal{A}_j.$$

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- A minimizer of the potential function is a pure strategy Nash equilibrium of a potential game (Monderer & Shapley, 1996).
- Convergence results in many learning algorithms (e.g., fictitious play, myopic learning, multiplicative updates) rely heavily on potential functions (Drighès et al., 2013; Fudenberg 1998; Marden et al., 2009; Monderer & Shapley, 1996)

Existence of a Potential Function

Lemma (Necessary Condition)

If the abstract game admits a potential function $V \in \mathcal{C}^2$, then

$$\sum_{e \in p_1 \cap p_2} \left[\frac{\partial}{\partial \phi_e^{\theta_i}} \tilde{\ell}_e^{\theta_j}((\phi_e^{\theta'})_{\theta' \in \Theta}) - \frac{\partial}{\partial \phi_e^{\theta_j}} \tilde{\ell}_e^{\theta_i}((\phi_e^{\theta'})_{\theta' \in \Theta}) \right] = 0,$$

for all $i, j \in [\![N]\!]$ *and* $p_1, p_2 \in \mathcal{P}$.

Existence of a Potential Function

Lemma (Sufficient Condition)

Assume that $|\Theta| = 2$. *If*

$$\sum_{e \in p_1 \cap p_2} \left[\frac{\partial}{\partial \phi_e^{\theta_1}} \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) - \frac{\partial}{\partial \phi_e^{\theta_2}} \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) \right] = 0,$$

for all $p_1, p_2 \in \mathcal{P}$, *then*

$$V((f_{p'}^{\theta_1})_{p' \in \mathcal{P}}, (f_{p'}^{\theta_2})_{p' \in \mathcal{P}}) = \sum_{e \in \mathcal{E}} \Bigg[\int_0^{\phi_e^{\theta_1}} \tilde{\ell}_e^{\theta_1}(u_1, \phi_e^{\theta_2}) \mathrm{d}u_1 + \int_0^{\phi_e^{\theta_2}} \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, u_2) \mathrm{d}u_2 - \int_0^{\phi_e^{\theta_2}} \int_0^{\phi_e^{\theta_1}} \frac{\partial}{\partial u} \tilde{\ell}_e^{\theta_1}(t, u) \mathrm{d}t \mathrm{d}u \Bigg]$$

is a potential function for the abstract game.

Finding a Nash Equilibrium

Theorem (Sufficient condition for Nash equilibria)

Assume that $|\Theta| = 2$, *and*

$$\sum_{e\in p_1\cap p_2}\left[\frac{\partial}{\partial\phi_e^{\theta_1}}\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},\phi_e^{\theta_2}) - \frac{\partial}{\partial\phi_e^{\theta_2}}\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2})\right] = 0,$$

for all $p_1, p_2 \in \mathcal{P}$. *If* $f = (f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta}$ *is a solution of the optimization problem*

$$\begin{aligned} \min\quad & V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}}, (f_{p'}^{\theta_2})_{p'\in\mathcal{P}}), \\ \text{s.t.}\quad & \sum_{p\in\mathcal{P}_k} f_p^{\theta_1} = F_k^{\theta_1} \text{ and } \sum_{p\in\mathcal{P}_k} f_p^{\theta_2} = F_k^{\theta_2},\ \forall k \in [\![K]\!], \\ & f_p^{\theta_1}, f_p^{\theta_2} \in \mathbb{R}_{\geq 0},\ \forall p \in \mathcal{P}, \end{aligned}$$

then $f = (f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta}$ *is a Nash equilibrium of the heterogeneous routing game.*

Characterizing All Nash Equilibria

Theorem

Furthermore, assume that potential function V is a convex function. Then $f = (f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}$ is a Nash equilibrium of the heterogeneous routing game if and only if it is a solution of the convex optimization problem

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- If finding a Nash equilibrium in the heterogeneous routing game is numerically intractable, it might be unlikely for the drivers to find it.

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- Driver of type $\theta \in \Theta$ must pay a toll $\tilde{\tau}_e^{\theta}((\phi_e^{\theta'})_{\theta' \in \Theta})$ for using an edge $e \in \mathcal{E}$.
- For using path $p \in \mathcal{P}_k$, she must pay $\tau_p^{\theta}(f) = \sum_{e \in p} \tilde{\tau}_e^{\theta}((\phi_e^{\theta'})_{\theta' \in \Theta})$.

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Nash Equilibrium

A flow vector $f = (f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}$ is a Nash equilibrium for the routing game with tolls if, for all $k \in [\![K]\!]$ and $\theta \in \Theta$, whenever $f_p^{\theta} > 0$ for some path $p \in \mathcal{P}_k$, then $\ell_p^{\theta}(f) + \tau_p^{\theta}(f) \leq \ell_{p'}^{\theta}(f) + \tau_{p'}^{\theta}(f)$ for all $p' \in \mathcal{P}_k$.

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- For using path $p \in \mathcal{P}_k$, she must pay $\tau_p^{\theta}(f) = \sum_{e \in p} \tilde{\tau}_e^{\theta}((\phi_e^{\theta'})_{\theta' \in \Theta})$.

Nash Equilibrium

A flow vector $f = (f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}$ is a Nash equilibrium for the routing game with tolls if, for all $k \in [\![K]\!]$ and $\theta \in \Theta$, whenever $f_p^{\theta} > 0$ for some path $p \in \mathcal{P}_k$, then $\ell_p^{\theta}(f) + \tau_p^{\theta}(f) \leq \ell_{p'}^{\theta}(f) + \tau_{p'}^{\theta}(f)$ for all $p' \in \mathcal{P}_k$.

- Consider the case that the tolls are type-independent (i.e., $\tilde{\tau}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) = \tilde{\tau}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) = \tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2})$) which is a harder case.

Imposing Tolls

Proposition

Assume that $|\Theta| = 2$. The abstract game admits the potential function

$$V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}}, (f_{p'}^{\theta_2})_{p'\in\mathcal{P}}) = \sum_{e\in\mathcal{E}} \Bigg[\int_0^{\phi_e^{\theta_1}} (\tilde{\ell}_e^{\theta_1}(u_1, \phi_e^{\theta_2}) + \tilde{\tau}_e(u_1, \phi_e^{\theta_2}))\mathrm{d}u_1 + \int_0^{\phi_e^{\theta_2}} (\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, u_2) + \tilde{\tau}_e(\phi_e^{\theta_1}, u_2))\mathrm{d}u_2 - \int_0^{\phi_e^{\theta_2}} \int_0^{\phi_e^{\theta_1}} \frac{\partial}{\partial u} (\tilde{\ell}_e^{\theta_1}(t, u) + \tilde{\tau}_e(t, u))\mathrm{d}t\mathrm{d}u \Bigg]$$

if

$$\frac{\partial \tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_2}} - \frac{\partial \tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_1}} = \frac{\partial \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_1}} - \frac{\partial \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_2}},$$

for all $e \in \mathcal{E}$.

Imposing Tolls

Corollary

Assume that $|\Theta| = 2$*. The abstract game admits a potential function* $V \in \mathcal{C}^2$ *if the imposed tolls are of the following from*

$$\tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2}) = c_e + \psi_e(\phi_e^{\theta_1} + \phi_e^{\theta_2}) + \int_0^{\phi_e^{\theta_2}} \left[\frac{\partial \tilde{\ell}_e^{\theta_2}(y,x)}{\partial y} - \frac{\partial \tilde{\ell}_e^{\theta_1}(y,x)}{\partial x} \right]_{x=q, y=\phi_e^{\theta_1}+\phi_e^{\theta_2}-q} \mathrm{d}q,$$

where $c \in \mathbb{R}_{\geq 0}$ *and* $\psi_e \in \mathcal{C}^1$ *are arbitrarily chosen for all* $e \in \mathcal{E}$*.*

Conclusions and Future Work

Conclusions

- Proved the existence of a Nash equilibrium in heterogeneous routing game.
- Characterized necessary and sufficient conditions for the existence of a potential function.
- Calculated tolls to ensure the existence of a potential function.

Future Work

- Study how heterogeneous populations can learn Nash equilibria;
- Extend the results to $|\Theta| > 2$, or a continuum of types.

Thank you.