1 Proof of Lemma 1

Lemma 1. Let \((\gamma_\tau)_{\tau \in \mathbb{N}}\) be a non-summable sequence of positive weights. If a real sequence \((u^{(\tau)})_{\tau \in \mathbb{N}}\) converges absolutely to \(u\) in the sense of Cesàro means w.r.t. \((\gamma_\tau)_{\tau}\), that is \(\lim_{T \to \infty} \frac{\sum_{\tau < T} \gamma_\tau |u^{(\tau)} - u|}{\sum_{\tau < T} \gamma_\tau} = 0\), then there exists a subset of indexes \(T\) of density one such that the subsequence \((u^{(\tau)})_{\tau \in T}\) converges to \(u\).

Proof. Let \(\epsilon > 0\), and define the set \(\mathcal{I}_\epsilon = \{\tau \in \mathbb{N} : \|u^{(\tau)} - u\| \geq \epsilon\}\). First, we show that for all \(\epsilon > 0\), \(\mathcal{I}_\epsilon\) has zero density. Indeed, we have for all \(\tau \in \mathcal{I}_\epsilon\), \(0 \leq \epsilon\), and thus

\[
0 \leq \sum_{\tau \in \mathcal{I}_\epsilon} \tau \epsilon \leq \sum_{\tau \in \mathbb{N} : \tau \leq T} \gamma_\tau \|u^{(\tau)} - u\| \sum_{\tau \in \mathbb{N} : \tau \leq T} \gamma_\tau
\]

But the right-hand side converges to 0 by assumption, therefore \(\lim_{T \to \infty} \frac{\sum_{\tau \in \mathbb{N} : \tau \leq T} \gamma_\tau}{\sum_{\tau \in \mathbb{N} : \tau \leq T} \gamma_\tau} = 0\), and \(\mathcal{I}_\epsilon\) has zero density w.r.t. \((\gamma_\tau)_{\tau}\).

Next, we will construct a set \(\mathcal{I} \subseteq \mathbb{N}\) of zero density, such that the subsequence \((u^{(\tau)})_{\tau \in \mathbb{N} \setminus \mathcal{I}}\) converges. For all \(k > 0\), let

\[
p_k(T) = \sum_{\tau \in \mathcal{I}_{\frac{k}{k+1}} : \tau \leq T} \gamma_\tau
\]

Since \(\lim_{T \to \infty} \frac{p_k(T)}{\sum_{\tau \in \mathbb{N} : \tau \leq T} \gamma_\tau} = 0\), \(\exists T_k > 0\) such that for all \(T \geq T_k\), \(\frac{p_k(T)}{\sum_{\tau \in \mathbb{N} : \tau \leq T} \gamma_\tau} \leq \frac{1}{k}\).

Without loss of generality, we can assume that \((T_k)_{k \in \mathbb{N}^*}\) is increasing. Now, let

\[
\mathcal{I} = \bigcup_{k \in \mathbb{N}^*} \left(\mathcal{I}_{\frac{k}{k+1}} \cap \{T_k, \ldots, T_{k+1} - 1\}\right).
\]

Then we have for all \(k \in \mathbb{N}^*\), \(\mathcal{I} \cap \{0, \ldots, T_{k+1} - 1\} = \left(\bigcup_{j=1}^k \mathcal{I}_{\frac{j}{j+1}}\right) \cap \{0, \ldots, T_{k+1} - 1\}\). But since \(\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{I}_k\), we have \(\mathcal{I} \cap \{0, \ldots, T_{k+1} - 1\} = \bigcup_{j=1}^k \mathcal{I}_{\frac{j}{j+1}} \cap \{0, \ldots, T_{k+1} - 1\}\), thus for all \(T\) such that \(T_k \leq T < T_{k+1}\), we have

\[
\frac{\sum_{\tau \in \mathcal{I} : \tau \leq T} \gamma_\tau}{\sum_{\tau \in \mathbb{N} : \tau \leq T} \gamma_\tau} \leq \frac{\sum_{\tau \in \mathcal{I}_{\frac{k}{k+1}} : \tau \leq T} \gamma_\tau}{\sum_{\tau \in \mathbb{N} : \tau \leq T} \gamma_\tau} = \frac{p_k(T)}{\sum_{\tau \in \mathbb{N} : \tau \leq T} \gamma_\tau} \leq \frac{1}{k}
\]

which proves that \(\mathcal{I}\) has zero density.

Let \(\mathcal{T} = \mathbb{N} \setminus \mathcal{I}\). We have that \(\mathcal{T}\) has density one, and it remains to prove that the subsequence \((u^{(\tau)})_{\tau \in T}\) converges to \(u\). For all \(k\), there exists \(S_k \in \mathcal{T}\) such that \(S_k \geq T_k\).

For all \(\tau \in \mathcal{T}\) with \(\tau \geq S_k\), there exists \(k' \geq k\) such that \(T_{k'} \leq \tau < T_{k'+1}\). Since \(\tau \notin \mathcal{I}\), we must have \(\tau \notin \mathcal{I}_{\frac{k'}{k'+1}}\), therefore

\[
\|u^{(\tau)} - u\| < \frac{1}{k'} \leq \frac{1}{k}.
\]

This proves that \((u^{(\tau)})_{\tau \in \mathcal{T}}\) converges to \(u\). \(\square\)
2 Proof of Lemma 3

Lemma 3 (Convergence of potentials under AREP algorithms). Let $\Gamma$ be a compact invariant set for the replicator ODE $\dot{\mu} = F(\mu)$, $V$ a Lyapunov function for $\Gamma$, and assume $V(\Gamma)$ has empty interior. Assume that the sequence of distributions $\{\mu^{(\tau)}\}_{\tau\in\mathbb{N}}$ obey an AREP update rule. Then the sequence of potentials $(V(\mu^{(\tau)}))_{\tau}$ converges.

Proof. Let $M$ be the affine interpolated process of the sequence $\{\mu^{(\tau)}\}_{\tau}$ with time steps $(\gamma^{(\tau)})_{\tau}$. That is, $M$ is the continuous function defined on $\mathbb{R}_{+}$ and with values in $\Delta$, such that for all $\tau$ and all $s \in [0, \gamma^{(\tau)})$

$$M(T^{(\tau)} + s) = \mu^{(\tau)} + s \frac{\mu^{(\tau+1)} - \mu^{(\tau)}}{\gamma^{(\tau)}}$$

where $T^{(\tau)} = \sum_{t \leq \tau} \gamma^{(t)}$.

First, by definition of the AREP class, we have $(\mu^{(\tau)})$ satisfies an update equation of the form

$$\mu^{(\tau+1)} - \mu^{(\tau)} = \gamma^{(\tau)} \left( F(\mu^{(\tau)}) + U^{(\tau+1)} \right)$$

where $(U^{(\tau)})$ is a bounded sequence of perturbations that satisfies the condition

$$\lim_{\tau_1 \to \infty} \max_{\tau_2, \sum_{t=t_1}^{t_2} \gamma^{(t)} \leq \tau} \left\| \sum_{t=t_1}^{t_2} \gamma^{(t)} U^{(t+1)} \right\| = 0$$

Furthermore, the sequence $(\mu^{(\tau)})$ is bounded since $\Delta$ is compact. By Proposition 4.1 in [1], the affine interpolated process $M$ is an asymptotic pseudo trajectory of the vector field $F$. It follows by Theorem 5.7 in [1] that the set $L(M)$ of limit points of $M$ is internally chain transitive.

Since $V$ is a Lyapunov function for the compact invariant set $\Gamma$, and $V(\Gamma)$ has empty interior, by Proposition 6.4 in [1], $\Gamma$ contains every internally chain transitive set and $V$ is constant on $\Gamma$. In particular, we have $L(M) \subseteq \Gamma$ and there exists $v \in \mathbb{R}$ such that $V(\mu) = v$ for all $\mu \in L(M)$.

Now consider the sequence of potential values $(V(\mu^{(\tau)}))_{\tau\in\mathbb{N}}$. Since $V$ is continuous and $\Delta$ is compact, $V(\Delta)$ is compact, and $(V(\mu^{(\tau)}))_{\tau\in\mathbb{N}}$ has at least one limit point. Let $\hat{v}$ be such a limit point, that is, $\hat{v}$ is the limit of a subsequence $(V(\mu^{(\tau)}))_{\tau\in\mathcal{T}}$ where $\mathcal{T} \subseteq \mathbb{N}$. The subsequence $(\mu^{(\tau)}),_{\tau\in\mathcal{T}}$ lives in the compact set $\Delta$, thus we can extract a further subsequence which converges, that is, there exists $\mathcal{T}' \subseteq \mathcal{T}$ and $\hat{\mu} \in \Delta$ such that $(\mu^{(\tau)})_{\tau\in\mathcal{T}'}$ converges to $\hat{\mu}$. By continuity of $V$, $V(\hat{\mu}) = \hat{v}$. But since $\hat{\mu} \in L(M)$, we also have $V(\hat{\mu}) = v$, therefore $\hat{v} = v$. This proves that $V(\mu^{(\tau)})$ has a unique limit point equal to $v$, thus it converges to $v$. \qed

References