STAT 205A - Problem Set 08

Walid Krichene (23265217)

November 2, 2013

(8.1) Suppose probability measures satisfy $\pi \ll \nu \ll \mu$. Show that

$$\frac{d\pi}{d\mu} = \frac{d\pi}{d\nu}\frac{d\nu}{d\mu}$$

proof Let $h_{\pi,\mu}$ be the Radon-Nikodym derivative of π with respect to μ, and similarly for the other measures. First, let us prove that for any integrable function f

$$\int f d\pi = \int h_{\pi,\mu} d\mu$$

this is true for indicator functions, since $\int 1_A d\pi = \pi(A) = \int_A h_{\pi,\mu} d\mu$ by definition of the Radon-Nikodym derivative. Then by linearity of the integral, the claim is also true for simple functions. Then we can show the claim for non-negative measurable functions, by approximating by simple functions. Indeed, if $f \geq 0$ is measurable, then there exists a sequence of simple functions $0 \leq s_n \uparrow f$ such that $\int s_n d\pi \uparrow \int f d\pi$. For all n, we have $\int s_n d\pi = \int s_n h_{\pi,\mu} = \int s_n h_{\pi,\mu} d\mu$ where $0 \leq s_n h_{\pi,\mu} \uparrow f h_{\pi,\mu}$ since $h_{\pi,\mu} \geq 0$. By the Monotone Convergence theorem, we have $\int f d\pi = \int f h_{\pi,\mu} d\mu$. Finally, to show the claim for any integrable function f, we can write $f = f^+ - f^-$ where each term is non-negative, and apply the previous result to each term.

Now we have for any integrable function f

$$\int f d\pi = \int f h_{\pi,\mu} d\mu \tag{1}$$

and applying the result to the measures $\pi \ll \nu$, we have

$$\int f d\pi = \int f h_{\pi,\nu} d\nu$$

where $f h_{\pi,\nu}$ is ν-integrable. Applying the result again to $\nu \ll \mu$, we have

$$\int f h_{\pi,\nu} d\nu = \int f h_{\pi,\nu} h_{\nu,\mu} d\mu \tag{2}$$

From equalities () and (2), we have for all integrable functions f

$$\int f(h_{\pi,\mu} - h_{\pi,\nu} h_{\nu,\mu}) d\mu = 0$$

in particular, taking $f = h_{\pi,\mu} - h_{\pi,\nu} h_{\nu,\mu}$ (this is an integrable function), we have

$$\int (h_{\pi,\mu} - h_{\pi,\nu} h_{\nu,\mu})^2 d\mu = 0$$

therefore $h_{\pi,\mu} - h_{\pi,\nu} h_{\nu,\mu} = 0$ a.e.

(indeed, if a non-negative function g is such that $\int g d\mu = 0$, then $\{\omega : g(\omega) > 0 = \cup_{n \geq 1} \{\omega : g(\omega) > \frac{1}{n}\}$, and by Markov's inequality, $\mu(\{\omega : g(\omega) > \frac{1}{n}\}) = P(g > \frac{1}{n}) \leq n \int g d\mu = 0$, thus the set $\{\omega : g(\omega) > 0\}$ is the countable union of null sets, and is a null set).

(8.2) In the setting of Theorem 7 [hard part], where S_2 is nice, show that Q is unique in the following sense. If Q^* is another conditional probability kernel for μ, then

$$\mu_1\{x : Q^*(x, B) = Q(x, B) \text{ for all } B \in \mathcal{S}_2\} = 1$$

proof First, by definition of a nice measure space, we can reduce the problem to the case where $S_2 = \mathbb{R}$ and $\mathcal{S}_2 = \mathcal{B}$ the collection of Borel sets.

First, we show the result for a countable generator of $\mathcal{B}$, namely the collection $\mathcal{C} = \{(-\infty, c), c \in \mathbb{Q}\}$. This is indeed a generator since for any real number a, there exists a non-decreasing sequence of rationals (c_n) such that $c_n \uparrow a$, thus $(-\infty, a) = \cup_{n\in\mathbb{N}}(\infty, c_n) \in \sigma(\mathcal{C})$.

Now consider the set

$$\begin{aligned} X &= \{x : Q^*(x, (-\infty, c)) = Q(x, (-\infty, c)) \text{ for all } c \in \mathbb{Q}\} \\ &= \cap_{c\in\mathbb{Q}} Q^*(x, (-\infty, c)) = Q(x, (-\infty, c))\} \end{aligned}$$

where we have by definition of the conditional probability Kernels, for any product-integrable function,

$$\int_{x\in S_1} \int_{y\in S_2} f(x, y)Q(x, dy)\mu_1(dx) = \int_{x\in S_1} \int_{S_2} f(x, y)Q^*(x, dy)\mu_1(dx)$$

Now fix c, and take $f(x, y) = [Q(x, (-\infty, c)) - Q^*(x, (-\infty, c))]1_{(-\infty,c)}(y)$. Then we have

$$\int_{x\in S_1} [Q(x, (-\infty, c)) - Q^*(x, (-\infty, c))]^2 \mu_1(dx) = 0$$

therefore

$$\mu_1(\{x : Q(x, (-\infty, c)) \neq Q^*(x, (-\infty, c))\}) = 0$$

and X^c is simply the countable union of such null sets, thus $\mu_1(X^c) = 0$, i.e. $\mu_1(X) = 1$.

This shows the result for a countable generator set. To show the result when $B \in \mathcal{S}_2$, it suffices to show that

$$X \subseteq \{x : Q(x, B) = Q^*(x, B) \text{ for all } B \in \mathcal{S}_2\}$$

let $x \in X$. We have $Q(x, \cdot)$ and $Q^*(x, \cdot)$ are both probability measures on S_2, and by definition of X, they agree on $\mathcal{C}$, which is a generator of the Borel σ-field $\mathcal{B}$, therefore they agree on $\mathcal{B}$ (by a familiar $\pi - \lambda$ argument), i.e. $Q(x, B) = Q^*(x, B)$ for all $B \in \mathcal{B}$. This proves the desired inclusion, and the result follows since

$$\mu_1\{x : Q^*(x, B) = Q(x, B) \text{ for all } B \in \mathcal{S}_2\} \geq \mu_1(X) = 1$$

(8.3) Let F be a distribution function. Let $c > 0$. Find a simple formula for

$$\int_{-\infty}^{\infty} (F(x+c) - F(x))dx$$

Claim:

$$\int_{-\infty}^{\infty} (F(x+c) - F(x))dx = c$$

proof Let μ be the probability measure associated to the distribution function F. Then we have for all x, $F(x) = \mu((-\infty, x]))$. Therefore we can write

$$\begin{aligned}
\int_{x\in\mathbb{R}} (F(x+c) - F(x))dx &= \int_{x\in\mathbb{R}} \int_{y\in\mathbb{R}} (1_{(y\le x+c)} - 1_{(y\le x)})d\mu(y)dx \\
&= \int_{x\in\mathbb{R}} \int_{y\in\mathbb{R}} 1_{(y\le x+c\ \&\ y>x)}d\mu(y)dx \\
&= \int_{y\in\mathbb{R}} \int_{x\in\mathbb{R}} 1_{(y\le x+c\ \&\ y>x)}dxd\mu(y) && \text{by Fubini's theorem} \\
&= \int_{y\in\mathbb{R}} \int_{x\in[y-c,y)} dxd\mu(y) \\
&= \int_{y\in\mathbb{R}} cd\mu(y) \\
&= c
\end{aligned}$$

which proves the claim.

(8.4) In the proof of Corollary 8, we used the inverse distribution function

$$f(x,u) = \inf\{y : u \le Q(x,(-\infty,y])\}$$

associated with the kernel Q. Show that f is product-measurable.

proof First, we show that f coincides with $\hat{f}$, defined as $\hat{f}(x,u) = \inf\{y \in \mathbb{Q} : u \le Q(x,(-\infty,y])\}$. We clearly have $f(x,u) \le \hat{f}(x,u)$ since the infimum is taken on a larger set for f. Thus it suffices to show that for all x,u, $f(x,u) \ge (\hat{f}(x,u))$. Fix x,u. We have by definition of $f(x,y)$ as the inf, for all $\epsilon > 0$, there exists $y < f(x,u)+\epsilon$ such that $u \le Q(x,(-\infty,y])$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $y' \in \mathbb{Q}$ such that $y \le y' \le f(x,u)+\epsilon$, and since $Q(x,\cdot)$ is a probability measure, we have

$$Q(x,(-\infty,y')) \ge Q(x,(-\infty,y)) \ge u$$

therefore $y' \ge \hat{f}(x,u)$. Combining the previous inequalities, we have

$$\hat{f}(x,y) \le y' \le f(x,y)+\epsilon$$

thus $\hat{f}(x,y) \le f(x,y)+\epsilon$. Since this holds for arbitrary $\epsilon > 0$, we have $\hat{f}(x,u) \le f(x,u)$. This proves the first claim.

Now to show that f (or equivalently $\hat{f}$) is product-measurable, it suffices to show that for all $c \in \mathbb{R}$, $\hat{f}^{-1}(c+\infty)$ is in the product σ-field. Fix $c \in \mathbb{R}$. We have

$$\begin{aligned}(x,u) \in \hat{f}^{-1}(c,+\infty) &\Leftrightarrow \inf\{y \in \mathbb{Q} : u \le Q(x,(-\infty,y])\} > c \\ &\Leftrightarrow \forall y \in \mathbb{Q} with y \le c,\ u > Q(x,(-\infty,y])) \\ &\Leftrightarrow \forall y \in \mathbb{Q} \cap [c,+\infty),\ g_y(x,u) > 0\end{aligned}$$

where $g_y(x,u) = u - Q(x,(-\infty,y]))$ is the difference of two product-measurable functions, thus is measurable. Therefore we can write

$$\hat{f}^{-1}(c,+\infty) = \cap_{y\in\mathbb{Q}\cap[c,+\infty)} g_y^{-1}(0,\infty)$$

which proves it is product-measurable as the countable intersection of product measurable sets $g_y^{-1}(0,+\infty)$. The proof is complete.

(8.5) Given a triple (X_1, X_2, X_3), we can define 3 probability measures $\mu_{12}, \mu_{13}, \mu_{23}$ on $\mathbb{R}^2$ by

$$\mu_{ij} \text{ is the distribution of } (X_i, X_j) \tag{3}$$

These probability measures satisfy a consistency condition: the marginal distribution μ_1 obtained from μ_{12} must coincide with the marginal distribution obtained from μ_{13}, and similarly for the marginal distributions μ_2 and μ_3.

Give an example to show that the converse is false. That is, give an example of $\mu_{12}, \mu_{13}, \mu_{23}$ that satisfy the consistency conditions, but for which there does not exists a triple (X_1, X_2, X_3) satisfying (3)

answer Assume Ω is finite, and consider the following joint distributions $\mu_{1,2} = \mu_{2,3} = \mu_{1,2}$ such that $\mu_{1,2}((x_1, x_2))$ is given by the following table

	0	1
0	0	$\frac{2}{5}$
1	$\frac{2}{5}$	$\frac{1}{5}$

(4)

where columns are values of x_1, and rows are values of x_2.

The consistency conditions are satisfied, since we have for example

$$\mu_{1,2}((0,0)) + \mu_{1,2}((0,1)) = \mu_{1,3}((0,0)) + \mu_{1,3}((0,1)) = \frac{2}{5}$$

and similarly for other conditions. This is by construction of $\mu_{i,j}$.

The claim is that there exists no random variables X_1, X_2, X_3 such that $\mu_{i,j}$ is the joint distribution of (X_i, X_j) for all i, j. Indeed, assume by contradiction that such variables exist, and let $\mu_{1,2,3}$ be their joint distribution. First, since Ω is finite, the image set $(X_1, X_2, X_3)(\Omega)$ is finite. Denote it by $\mathcal{X}$. Then we can write

$$\mu_{1,2,3}(B) = \sum_{\xi \in \mathcal{X}} \mu_{1,2,3}(\xi) 1_{(\xi \in B)}$$

where $a_\xi \geq 0$. Next, we argue that $a_\xi = 0$ for all $\xi \notin \{0,1\}^3$. Indeed, if $(x_1, x_2) \notin \{0,1\}^2$, then $\mu_{1,2}(x_1, x_2) = 0$ by definition of $\mu_{1,2}$, thus $\sum_{x_3} \mu_{1,2,3}(x_1, x_2, x_3) = 0$. Similarly for $(x_2, x_3) \notin \{0,1\}^2$ and $(x_1, x_3) \notin \{0,1\}^2$. Now we can write the marginal distribution conditions

$$\mu_{1,2,3}(0,0,0) + \mu_{1,2,3}(0,0,1) = \mu_{1,2}(0,0) = 0$$
$$\mu_{1,2,3}(0,1,0) + \mu_{1,2,3}(0,1,1) = \mu_{1,2}(0,1) = \frac{2}{5}$$
$$\mu_{1,2,3}(1,0,0) + \mu_{1,2,3}(1,0,1) = \mu_{1,2}(1,0) = \frac{2}{5}$$
$$\mu_{1,2,3}(1,1,0) + \mu_{1,2,3}(1,1,1) = \mu_{1,2}(1,1) = \frac{1}{5}$$

$$\mu_{1,2,3}(0,0,0) + \mu_{1,2,3}(0,1,0) = \mu_{1,3}(0,0) = 0$$
$$\mu_{1,2,3}(0,0,1) + \mu_{1,2,3}(0,1,1) = \mu_{1,3}(0,1) = \frac{2}{5}$$
$$\mu_{1,2,3}(1,0,0) + \mu_{1,2,3}(1,1,0) = \mu_{1,3}(1,0) = \frac{2}{5}$$
$$\mu_{1,2,3}(1,0,1) + \mu_{1,2,3}(1,1,1) = \mu_{1,3}(1,1) = \frac{1}{5}$$

$$\mu_{1,2,3}(0,0,0) + \mu_{1,2,3}(1,0,0) = \mu_{2,3}(0,0) = 0$$
$$\mu_{1,2,3}(0,0,1) + \mu_{1,2,3}(1,0,1) = \mu_{2,3}(0,1) = \frac{2}{5}$$
$$\mu_{1,2,3}(0,1,0) + \mu_{1,2,3}(1,1,0) = \mu_{2,3}(1,0) = \frac{2}{5}$$
$$\mu_{1,2,3}(0,1,1) + \mu_{1,2,3}(1,1,1) = \mu_{2,3}(1,1) = \frac{1}{5}$$

then

$$\mu_{1,2,3}(0,0,0) = \mu_{1,2,3}(0,0,1) = \mu_{1,2,3}(0,1,0) = \mu_{1,2,3}(1,0,0) = 0$$
$$\mu_{1,2,3}(0,1,1) = \mu_{1,2,3}(1,0,1) = \mu_{1,2,3}(1,1,0) = \frac{2}{5}$$
$$\mu_{1,2,3}(1,1,0) + \mu_{1,2,3}(1,1,1) = \mu_{1,2,3}(1,0,1) + \mu_{1,2,3}(1,1,1) = \mu_{1,2,3}(0,1,1) + \mu_{1,2,3}(1,1,1) = \frac{1}{5}$$

which implies that $\mu_{1,2,3}(1,1,1) = -\frac{1}{5}$, contradiction. This provides a counter-example.