STAT 205A - Problem Set 05

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(5.1) Let (X_n) be i.i.d. with $\mathrm{E}\,|X| < \infty$. Let $M_n = \max(X_1, \dots, X_n)$. Prove that $\frac{1}{n} M_n \to 0$ a.s.

proof Let $S_n = \sum_{i=1}^n |X_i|$. By the SLLN, we have $\frac{S_n}{n} \to E|X|$ a.s. Now we have

$$\frac{|X_n|}{n} = \frac{S_n}{n} - \frac{S_{n-1}}{n} = \frac{S_n}{n} - \frac{S_{n-1}}{n-1}\frac{n-1}{n}$$

and since $\lim_{n\to\infty} \frac{S_n}{n} = \mathrm{E}\,|X|$ a.s. and $\frac{n-1}{n} = 1$ a.s., we have with probability 1, $\frac{|X_n|}{n}$ converges and its limit is 0. Now for all $\omega \in \Omega$ such that $\frac{|X_n(\omega)|}{n} \to 0$, we have

$$|M_n(\omega)| \leq \max(|X_1(\omega)|, \dots, |X_n(\omega)|)$$

therefore taking the limsup

$$\limsup_n \frac{|M_n|}{n} \leq \limsup \frac{\max(|X_1(\omega)|, \dots, |X_n(\omega)|)}{n} = \limsup_n \frac{|X_n(\omega)|}{n} = 0$$

thus $\frac{M_n(\omega)}{n}$ converges to 0. This proves that $\frac{M_n}{n} \to 0$ a.s.

(5.2) Let $0 \leq X_1 \leq X_2 \leq \dots$ be random variables such that $\mathrm{E}\,X_n \sim an^\alpha$ and $\mathrm{var}(X_n) \leq Bn^\beta$, where $B < \infty$, and $0 < \beta < 2\alpha < \infty$. Prove that $n^{-\alpha} X_n \to a$ a.s.

proof Since $\mathrm{E}\,X_n \sim an^\alpha$, there exists N such that for all $n > N$, $\mathrm{E}\,X_n > 0$. Without loss of generality, we restrict the sequence to the terms $(X_n)_{n>N}$. Let $Y_n = \frac{X_n}{\mathrm{E}\,X_n}$. Then $\mathrm{E}\,Y_n = 1$, $\mathrm{var} = \frac{\mathrm{var}\,X_n}{(\mathrm{E}\,X_n)^2}$, and to prove the result, it suffices to show that $Y_n \to 1$ a.s. Indeed, if $Y_n \to 1$ a.s., then $\frac{X_n}{n^\alpha} = aY_n \frac{\mathrm{E}\,X_n}{an^\alpha}$, and this converges to a whenever Y_n converges to 1.

Let $\epsilon > 0$. Using Chebyshev's inequality, we can bound deviations of Y_n

$$P(|Y_n - 1| \geq \epsilon) \leq \frac{\mathrm{var}\,Y_n}{\epsilon^2} = \frac{\mathrm{var}\,X_n}{\epsilon^2(\mathrm{E}\,X_n)^2} \leq \frac{Bn^\beta}{\epsilon^2(\mathrm{E}\,X_n)^2} \sim \frac{B}{\epsilon^2 a}\frac{1}{n^{2\alpha-\beta}}$$

where $2\alpha - \beta > 0$ by assumption.

If we consider a subsequence $(Y_{n_j})_j$, the deviations of the subsequence are then given by

$$P(|Y_{n_j} - 1| \geq \epsilon) \leq\sim \frac{B}{\epsilon^2 a}\frac{1}{n_j^{2\alpha-\beta}}$$

in particular, if we choose $n_j = j^\gamma$ where γ is an integer and $\gamma > \frac{1}{2\alpha-\beta}$, then the upper bound become summable since

$$\frac{1}{n_j^{2\alpha-\beta}} = \frac{1}{j^{\gamma(2\alpha-\beta)}}$$

and $\gamma(2\alpha - \beta) > 1$.

Therefore by the first Borel-Cantelli Lemma, $P(|Y_{n_j} - 1| \geq \epsilon \text{ infinitely often}) = 0$. Therefore almost surely, $|Y_n - 1| < \epsilon$ ultimately. And since this is true for all ϵ, we have $Y_{n_j} \to 1$ a.s.

Now consider ω such that $Y_{n_j} \to 1$ a.s. We have for all $n_j \leq n \leq n_{j+1}$,

$$\frac{X_{n_j}(\omega)}{\mathrm{E}\, X_{n_{j+1}}} \leq \frac{X_n(\omega)}{\mathrm{E}\, X_n} \leq \frac{X_{n_{j+1}}(\omega)}{\mathrm{E}\, X_{n_j}}$$

which can be rewritten as

$$\frac{\mathrm{E}\, X_{n_j}}{\mathrm{E}\, X_{n_{j+1}}} Y_{n_j}(\omega) \leq Y_n(\omega) \leq \frac{\mathrm{E}\, X_{n_{j+1}}}{\mathrm{E}\, X_{n_j}} Y_{n_{j+1}}(\omega)$$

both bounds converge to 1, as $j \to \infty$ (by assumption on the expectation, $\frac{\mathrm{E}\, X_{n_j}}{\mathrm{E}\, X_{n_{j+1}}} \sim 1$, and by choice of ω, $Y_{n_j}(\omega) \to 0$). This shows that for all such ω, $Y_n(\omega) \to 1$. This proves that whenever $Y_{n_j} \to 1$, then $Y_n \to 1$, which concludes the proof.

(5.3) Prove that the following are equivalent

(i) $X_n \to X$ in probability

(ii) There exists a sequence $\epsilon_n \downarrow 0$ such that $P(|X_n - X| > \epsilon_n) \leq \epsilon_n$ for all n.

(iii) $\mathrm{E} \min(X_n - X, 1) \to 0$

proof

- (i) $\Rightarrow$ (ii): we have for all $\epsilon > 0$, $P(|X_n - X| > \epsilon) \to 0$ as $n \to \infty$. In particular, taking $\epsilon = 1/k$, $k \geq 1$, there exits N_k such that for all $n > N_k$, $P(|X_n - X| > 1/k) \leq 1/k$. Without loss of generality, we can assume the sequence N_k non-decreasing. Now define the sequence ϵ_n as follows
  - for all $n \leq N_1$, $\epsilon_n = 1$. Then for all $n \leq N_1$, we do have $P(|X_n - X| > \epsilon_n) \leq 1 = \epsilon_n$.
  - for all $k \geq 1$, for all $n \in (N_k, N_{k+1}]$, $\epsilon_n = \frac{1}{k}$. Then we have by definition of N_k for all $n \in (N_k, N_{k+1})$, $P(|X_n - X| > \epsilon_n) \leq \frac{1}{k} = \epsilon_n$.
- (ii) $\Rightarrow$ (i): Let $\epsilon > 0$. Since $\epsilon_n \downarrow 0$, there exists N such that for all $n > N$, $\epsilon_n < \epsilon$. For all $n > N$, we have $|X_n - X| > \epsilon \Rightarrow |X_n - X| > \epsilon_n$, therefore $P(|X_n - X| > \epsilon) \leq P(|X_n - X| \leq \epsilon_n) \leq \epsilon_n$, which converges to 0 as $n \to \infty$. This shows $X_n \to X$ in probability.
- (i) $\Rightarrow$ (iii): let $\epsilon > 0$. Writing the expectation as an integral, we can partition $\Omega = A \cup A^c$, where $A = \{\omega : \min(|X_n - X|, 1) > \epsilon/2\}$. Then

$$\begin{aligned}\mathrm{E} \min(|X_n - X|, 1) &= \int_\Omega \min(|X_n - X|, 1) d\mu \\ &= \int_A \min(|X_n - X|, 1) d\mu + \int_{A^c} \min(|X_n - X|, 1) d\mu\end{aligned}$$

  for all $\omega \in A^c$, we have $\min(X_n(\omega) - X(\omega), 1) \leq \epsilon/2$, therefore $\int_{A^c} \min(|X_n - X|, 1) d\mu \leq \epsilon/2$.

  To bound the second term in the sum, we use convergence in probability: for all $\omega \in A$, we have $\min(X_n(\omega) - X(\omega), 1) \leq 1$, thus $\int_A \min(|X_n - X|, 1) d\mu \leq \mu(A) = P(|X_n - X| > \epsilon/2)$ which converges to 0 by assumption. Therefore there exists N such that for all $n > N$, $\int_A \min(|X_n - X|, 1) d\mu \leq \epsilon/2$. Combining the two bounds, we have for all $n > N$, $\mathrm{E} \min(|X_n - X|, 1) \leq \epsilon$. Since this is true for all $\epsilon > 0$, we have the desired result.

- (iii) ⇒ (i): Let $0 < \epsilon < 1$. Consider the function $\phi(x) = \min(x, 1)$, defined on $\mathbb{R}_+$. We have by Markov's inequality

$$P(\phi(|X_n - X|) > \epsilon) \leq \frac{\mathrm{E}\min(|X_n - X|, 1)}{\phi(\epsilon)} = \frac{\mathrm{E}\min(|X_n - X|, 1)}{\epsilon}$$

which converges to 0 by assumption. This concludes the proof.

(5.4. Investment problem) Assume that at the beginning of each year, you can buy bonds for \$1 that are worth \$$a$ at the end of the year, or stocks that are worth a random amount $V_n \geq 0$. If you always invest a fixed proportion p of your wealth in bonds, then your wealth at the end of year $n+1$ is

$$W_{n+1} = W_n(ap + (1-p)V_n)$$

where $a \geq 1$. Suppose $V_1, V_2, \ldots$ are i.i.d. with $\mathrm{E}\,V_n^2 < \infty$ and $\mathrm{E}\,\frac{1}{V_n^2} < \infty$

(i) Show that $\frac{\log W_n}{n} \to c(p)$ a.s., where $c(p)$ is a function of p

proof We first observe that

- by convexity of $x \mapsto x^2$, $(\mathrm{E}\,V)^2 < \mathrm{E}\,V^2 < \infty$ thus $\mathrm{E}\,V < (\mathrm{E}\,V^2)^{\frac{1}{2}} < \infty$
- by convexity of $x \mapsto x^{-2}$, $\frac{1}{(\mathrm{E}\,V)^2} < \mathrm{E}\,\frac{1}{V^2} < \infty$ thus $\mathrm{E}\,V > \frac{1}{(\mathrm{E}\,\frac{1}{V^2})^{\frac{1}{2}}} > 0$

Let w_1 be the initial wealth. The we have for all n,

$$\frac{\log W_n}{n} = \frac{w_1 + \sum_{i=1}^n \log(ap + (1-p)V_n)}{n}$$

where the first term $\frac{w_1}{n}$ is deterministic and converges to 0. The random variables $\log(ap+(1-p)V_n)$ are i.i.d. To show that they have finite expectation, we first have the following upper bound by concavity of the log:

$$\mathrm{E}\log(ap + (1-p)V) \leq \log \mathrm{E}(1 + ap + (1-p)V) = \log(1 + ap + (1-p)\,\mathrm{E}\,V) < \infty$$

and to obtain a lower bound, we distinguish two cases: if $p > 0$, then $\log(ap + (1-p)V) \geq \log(p)$ and taking the expectation we have $\mathrm{E}\log(ap + (1-p)V) > \log(p) > -\infty$. If $p = 0$, then we need to check $\mathrm{E}\log V > -\infty$. This is obtained by writing

$$\begin{aligned}
\mathrm{E}\log V &\geq \mathrm{E}\,1_{V<1}\log V \\
&\geq \mathrm{E}\,1_{V<1}\left(\frac{1}{2} - \frac{1}{2V^2}\right) && \text{since for all } v < 1,\ \frac{1}{2} - \frac{1}{2v^2} \leq \log v \\
&> -\infty && \text{since } \mathrm{E}\,\frac{1}{V^2} < \infty
\end{aligned}$$

Therefore by the SLLN

$$\frac{\sum_{i=1^n} \log(ap + (1-p)V_n)}{n} \to \mathrm{E}\log(ap + (1-p)V) \text{ a.s.}$$

which shows that

$$\frac{\log W_n}{n} \to \mathrm{E}\log(ap + (1-p)V) \text{ a.s.}$$

(ii) Show that $c(p)$ is concave

proof Let $c(p) = \mathrm{E}\log(ap + (1-p)V)$, and let $l(\omega)(p) = ap + (1-p)V(\omega)$, a linear function in p. We have for all $p, p' \in [0,1]$, and for all $\lambda \in [0,1]$, and all ω,

$$\begin{aligned} \log l(\omega)(\lambda p + (1-\lambda)p') &= \log \lambda l(\omega)(p) + (1-\lambda)l(\omega)(p') && \text{by linearity of } e(\omega)(\cdot) \\ &\geq \lambda \log l(\omega)(p) + (1-\lambda)\log l(\omega)(p') && \text{by Jensen's inequality} \end{aligned}$$

then taking the expectation, we have

$$c(\lambda p + (1-\lambda)p') \geq \lambda c(p) + (1-\lambda)c(p')$$

i.e. $c(\cdot)$ is concave.

(iii) By investigating $c'(0)$ and $c'(1)$, give sufficient conditions on V that guarantee that the optimal choice of p is in $(0,1)$

proof Assuming the conditions of the derivation under the expectation theorem hold, we have

$$c'(p) = \mathrm{E}\,\frac{a-V}{ap + (1-p)V}$$

thus

$$\begin{aligned} c'(0) &= \mathrm{E}\,\frac{a-V}{V} \\ c'(1) &= \mathrm{E}\,\frac{a-V}{a} \end{aligned}$$

and we have $c'(p)$ is non-increasing since c is concave, thus $c'(0) \geq c'(1)$. To have the expectation minimal in the open interval $p \in (0,1)$, it suffices that the derivative vanishes inside $(0,1)$, i.e. $c'(0) < 0 < c'(1)$, i.e.

$$\mathrm{E}\,\frac{a-V}{V} < 0 < \frac{a-V}{a}$$

which we can rewrite as

$$\begin{aligned} \mathrm{E}\,\frac{1}{V} &> \frac{1}{a} \\ \mathrm{E}\,V &> a \end{aligned}$$

(iv) Suppose $P(V=1) = P(V=4) = \frac{1}{2}$. Find the optimal p as a function of a.

answer Under this assumption, we have

$$\begin{aligned} \mathrm{E}\,V &= \frac{1}{2}(1+4) = \frac{5}{2} \\ \mathrm{E}\,\frac{1}{V} &= \frac{1}{2}(1+\frac{1}{4}) = \frac{5}{8} \end{aligned}$$

and we have by (iii) the following cases:

(a) if $a \geq \mathrm{E}\,V = \frac{5}{2}$, then $c'(1) \geq 0$, and $p = 1$ is optimal (buy all bonds).

(b) if $a \leq \frac{1}{\mathrm{E}\,\frac{1}{V}} = \frac{8}{5}$, then $c'(0) \leq 0$, and $p = 0$ is optimal (buy all stock).

(c) if $a \in (\frac{8}{5}, \frac{5}{2})$, then $c'(0) < 0 < c'(1)$ and the optimal p is strictly in $(0,1)$. In this case we can compute the derivative of c:

$$c'(p) = \frac{1}{2}\left(\frac{a-1}{ap+(1-p)} + \frac{a-4}{ap+4(1-p)}\right)$$

and $c'(p) = 0$ if and only if

$$\frac{a-1}{ap+(1-p)} + \frac{a-4}{ap+4(1-p)} = 0$$

i.e.

$$(a-1)(ap+4(1-p)) + (a-4)(ap+(1-p)) = 0$$

i.e.

$$2p(a-4)(a-1) + 5a - 8 = 0$$

i.e.

$$p(a) = \frac{8-5a}{2(a-4)(a-1)}$$

note that this expression is increasing in a, as one expects (if the value of bond increases), $p(a) \in (0,1)$ whenever $a \in (\frac{5}{8}, \frac{5}{2})$.

(5.5) Prove the deterministic lemma we used in the proof of the Glivenko-Cantelli Theorem

Lemma 1 *If $F_1, F_2, \ldots$ and F are distribution functions and*

(i) $F_n(x) \to F(x)$ for all $x \in \mathbb{Q}$

(ii) $F_n(x) \to F(x)$ and $F_n(x-) \to F(x-)$ for each atom x of F

then F_n converges uniformly to F.

proof We first show that $F_n(x) \to F(x)$ and $F_n(x-) \to F(x-)$ for all $x \in \mathbb{R}$. This is true for jump points by assumption. Consider $x \in \mathbb{R}$ such that F is continuous at x. Let $p, q \in \mathbb{Q}$ such that $p < x < q$. Then for all n, $F_n(p) \le F_n(x) \le F_n(q)$, and taking limits,

$$F(p) \le \liminf_n F_n(x) \le \limsup_n F_n(x) \le F(q)$$

since $\lim_{p\uparrow x} F(p) = \lim_{p\downarrow x} F(q) = F(x)$, it follows that $\liminf F_n(x) = \limsup F_n(x) = F(x)$, which proves $F_n(x) \to F(x)$.

To prove the result for $x-$, we have for all n,

$$F_n(x-) \le F_n(x)$$

thus

$$\begin{aligned} \limsup_n F_n(x-) \le \limsup_n F_n(x) \qquad & = F(x) \text{by the previous part} \\ = F(x-) \qquad & \text{by continuity} \end{aligned}$$

now let $q \in \mathbb{Q}$ with $q < x$. We have $F_n(q) \le F_n(x-)$ thus taking the limit

$$F(q) \le \liminf F_n(x-)$$

and letting $q \uparrow x-$

$$F(x-) \leq \liminf F_n(x-)$$

which proves the result.

Now we have $F_n(x) \to F(x)$ and $F_n(x-) \to F(x-)$ for all $x \in \mathbb{R}$. We next show uniform convergence.

Let $\epsilon > 0$. There exists a finite family of points in $\mathbb{R}$, $x_1 < \cdots < x_k$ such that

$$\begin{aligned}
&F(x_{i+1}^-) - F(x_i) \leq \epsilon \text{ for all } i \\
&F(x_1) - 0 \leq \epsilon \\
&1 - F(x_k) \leq \epsilon
\end{aligned}$$

one can construct the sequence by induction, simply by

- $x_1 = \sup\{x : F(x) \leq \epsilon\}$
- $x_{i+1} = \sup\{x : F(x) \leq \epsilon + F(x_i)\}$

Then for each i, $F(x_{i+1}-) - F(x_i) \leq \epsilon$ as required. The process will terminate after finitely many iterations since we observe that for each i $F(x_{i+1}) - F(x_i) \geq \epsilon$ (otherwise $F(x_{i+1}) < F(x_i) + \epsilon$, and by right continuity there exists $x > x_{i+1}$ such that $F(x) < F(x_i) + \epsilon$, which contradict the definition of x_{i+1}.

Now for all $x \in \mathbb{R}$, there exists i such that $x_i \leq x \leq x_{i+1}$, thus

$$\begin{aligned}
&F_n(x_i) \leq F_n(x) \leq F_n(x_{i+1}-) \\
&F(x_i) \leq F(x) \leq F(x_{i+1}-)
\end{aligned}$$

thus

$$\begin{aligned}
F_n(x) - F(x) &\geq F_n(x_i) - F(x_{i+1}-) \\
&= F_n(x_i) - F(x_i) + F(x_i) - F(x_{i+1}-) \\
&\geq F_n(x_i) - F(x_i) - \epsilon
\end{aligned}$$

and

$$\begin{aligned}
F_n(x) - F(x) &\leq F_n(x_{i+1}-) - F(x_i) \\
&= F_n(x_{i+1}-) - F(x_{i+1}-) + F(x_{i+1}-) - F(x_i) \\
&\leq F_n(x_{i+1}-) - F(x_{i+1}-) + \epsilon
\end{aligned}$$

and since we proved convergence of $F_n(x) \to F(x)$ and $F_n(x-) \to F(x-)$ for all $x \in \mathbb{R}$, for n large enough we have

$$\begin{aligned}
&F_n(x_i) - F(x_i) \geq -\epsilon \\
&F_n(x_{i+1}-) - F(x_{i+1}-) \leq \epsilon
\end{aligned}$$

for all x_i (there are finitely many such conditions). Therefore for n large enough

$$\begin{aligned}
&F_n(x) - F(x) \geq -2\epsilon \\
&F_n(x) - F(x) \leq 2\epsilon
\end{aligned}$$

which concludes the proof.