(4.1. Monte Carlo Integration) Let $f : [0, 1] \rightarrow \mathbb{R}$ be such that $\int_{[0,1]} f^2(x) \, dx < \infty$. Let $U_i$ be i.i.d. Uniform$(0, 1)$. Let

$$D_n = n^{-1} \sum_{i=1}^{n} f(U_i) - \int_{0}^{1} f(x) \, dx$$

1. Use Chebyshev’s inequality to bound $P(|D_n| > \epsilon)$

**answer** The density function of $U$ is simply the indicator function of $[0, 1]$. Therefore the expectation of $D_n$ is given by

$$E \, D_n = n^{-1} \sum_{i=1}^{n} \int_{u=0}^{1} f(u) \, du - \int_{x=0}^{1} f(x) \, dx = 0$$

Now by Chebyshev’s inequality, we have

$$P(|D_n| > \epsilon) \leq \frac{1}{\epsilon^2} \text{var}(D_n)$$

writing $\int_{0}^{1} f(x) \, dx = E \, f(I)$, we have

$$\text{var}(D_n) = E \left( \frac{1}{n} \sum_{i=1}^{n} f(U_i) - E \, f(U) \right)^2$$

$$= \frac{1}{n^2} E \left( \sum_{i=1}^{n} f(U_i) \right)^2 + (E \, f(U))^2 - \frac{2}{n} E \, f(U) \sum_{i=1}^{n} E \, f(U_i)$$

$$= \frac{1}{n^2} \left( \sum_{i=1}^{n} E \, f^2(U_i) + 2 \sum_{1 \leq i < j \leq n} E(f(U_i)f(U_j)) \right) + (E \, f(U))^2 - 2 (E \, f(U))^2$$

$$= \frac{1}{n^2} \left( \sum_{i=1}^{n} E \, f^2(U_i) + 2 \sum_{1 \leq i < j \leq n} E(U_i) E(f(U_j)) \right) - (E \, f(U))^2$$

by independence of $f(U_i)$’s

$$= \frac{1}{n} E \, f^2(U) + \frac{2n(n-1)}{n^2} (E \, f(U))^2 - (E \, f(U))^2$$

$$= \frac{1}{n} E \, f^2(U) + \frac{n-2}{n} (E \, f(U))^2$$

which written in integral form, is

$$\text{var}(D_n) = \frac{1}{n} \int_{u=0}^{1} f^2(u) \, du + \frac{n-2}{n} \left( \int_{0}^{1} f(u) \, du \right)^2$$

2. Show this bound remains true if the $U_i$ are only pairwise independent.
**proof**  Pairwise independence is sufficient since in the above proof, it suffices that $E f(U_i) f(U_j) = E f(U_i) E f(U_j)$ for all $i,j$, which is true when $U_i$'s are pairwise independent.

(4.2) Let $X \geq 0$ and $Y \geq 0$ be independent r.v.'s with densities $f_X$ and $f_Y$. Calculate the densities of $XY$ and $X/Y$.

**answer**

- Let $f_{XY}$ be the density of $XY$. Then for all $c \in \mathbb{R}$,
  
  $$P(XY \leq c) = \int_{u=0}^{c} f_{XY}(u) du$$

  but we can also write
  
  $$P(XY \leq c) = \int_{x \in \mathbb{R}_+} \int_{y \in \mathbb{R}_+} 1_{xy \leq c} f_X(x) f_Y(y) dxdy$$

  $$= \int_{x \in \mathbb{R}_+ \setminus \{0\}} \left( \int_{y=0}^{c/x} f_Y(y) dy \right) f_X(x) dx$$

  since $\{0\}$ has measure 0

  $$= \int_{x \in \mathbb{R}_+ \setminus \{0\}} \left( \int_{u=c}^{0} f_Y(u/x) du \right) f_X(x) dx$$

  using the change of variable $u = xy$

  $$= \int_{u=0}^{c} \left( \int_{x \in \mathbb{R}_+ \setminus \{0\}} 1_{x/y \leq c} f_Y(u/x) f_X(x) dx \right) du$$

  by Fubini

  therefore we have for all $c$,
  
  $$\int_{u=0}^{c} f_{XY}(u) du = \int_{u=0}^{c} \left( \int_{x \in \mathbb{R}_+ \setminus \{0\}} \frac{1}{x} f_Y(u/x) f_X(x) dx \right) du$$

  and taking derivative with respect to $c$ on both sides, we conclude that
  
  $$f_{XY}(u) = \left( \int_{x \in \mathbb{R}_+ \setminus \{0\}} \frac{1}{x} f_Y(u/x) f_X(x) dx \right)$$

- Let $f_{X/Y}$ be the density of $X/Y$. Then for all $c \in \mathbb{R}$,
  
  $$P(X/Y \leq c) = \int_{u=0}^{c} f_{X/Y}(u) du$$

  but we can also write
  
  $$P(X/Y \leq c) = \int_{x \in \mathbb{R}_+, y \in \mathbb{R}_+ \setminus \{0\}} 1_{x/y \leq c} f_X(x) f_Y(y) dxdy$$

  since $X$ and $Y$ are independent

  $$= \int_{y \in \mathbb{R}_+ \setminus \{0\}} \left( \int_{x=0}^{yc} f_X(x) dx \right) f_Y(y) dy$$

  B Fubini

  $$= \int_{y \in \mathbb{R}_+ \setminus \{0\}} \left( \int_{u=0}^{c} y f_X(uy) du \right) f_Y(y) dy$$

  using the change of variable $u = x/y$

  $$= \int_{u=0}^{c} \left( \int_{y \in \mathbb{R}_+ \setminus \{0\}} y f_X(uy) f_Y(y) dy \right) du$$

  by Fubini

2
therefore we have for all $c$,
\[
\int_{u=0}^{c} f_{X/Y}(u) du = \int_{u=0}^{c} \left( \int_{y \in \mathbb{R}_+ \setminus \{0\}} y f_X(uy) f_Y(y) dy \right) du
\]
and taking derivative with respect to $c$ on both sides, we conclude that
\[
f_{X/Y}(u) = \left( \int_{y \in \mathbb{R}_+ \setminus \{0\}} y f_X(uy) f_Y(y) dy \right)
\]

(4.3) Let $(X_i)$ be r.v.’s with $E X_i = 0$ and $E X_i X_j \leq r(j - i)$ for $1 \leq i \leq j < \infty$, where $r(n)$ is a deterministic sequence with $r(n) \to 0$ as $n \to \infty$. Prove that $n^{-1} \sum_{i=1}^{n} X_i \to 0$ in Probability.

**proof** Let $S_n = \sum_{i=1}^{n} X_i$. By Chebyshev’s inequality, we have
\[
P\left(\frac{|S_n|}{n} > \epsilon\right) \leq \frac{1}{n^2 \epsilon^2} E[S_n^2]
\leq \frac{1}{n^2 \epsilon^2} \sum_{i=1}^{n} \sum_{j=1}^{n} E[X_i X_j]
\leq \frac{1}{n^2 \epsilon^2} \sum_{i=1}^{n} \sum_{j=1}^{n} r(|j - i|)
\]
Now for any fixed $i$,
\[
\sum_{j=1}^{n} r(|j - i|) = \sum_{j=1}^{i} r(i - j) + \sum_{j=i+1}^{n} r(j - i) = \sum_{k=0}^{i-1} r(k) + \sum_{l=1}^{n-i} r(l) \leq 2 \sum_{k=1}^{n} |r(k)|
\]
Therefore
\[
P\left(\frac{|S_n|}{n} > \epsilon\right) \leq \frac{1}{n^2 \epsilon^2} 2n \sum_{k=1}^{n} |r(k)|
= \frac{2 \sum_{k=1}^{n} |r(k)|}{n}
\]
and since $(r(n))_n$ converges to 0, so does $(|r(n)|)_n$, and so does the sequence of Cesàro means $(\sum_{k=1}^{n} |r(k)|)_n$ (for any $\delta > 0$, there exists $N$ such that for all $n > N$, $|r_n| < \delta$. Then we can write $\frac{1}{n} \sum_{k=1}^{n} |r_k| = \frac{1}{n} \sum_{k=1}^{N} |r_k| + \frac{n-N}{n} \delta \leq \frac{\sum_{k=1}^{N} |r_k|}{n} + \delta \leq 2\delta$ for $n$ large enough).

Therefore $P\left(\frac{|S_n|}{n} > \epsilon\right)$ converges to 0 as $n \to \infty$

(4.4) Suppose events $A_n$ satisfy $P(A_n) \to 0$ and
\[
\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty
\]
Prove that $P(A_n$ infinitely often) = 0
proof  Let us write \( B_n = A_{n+1} \setminus A_n = A_{n+1} \cap A_n^c \).

By the first Borel-Cantelli Lemma, applied to the events \( B_n \), we have \( P(B_n \text{ infinitely often}) = 0 \), i.e.

\[
P(\bigcap_{m=1}^{\infty} \cup_{n \geq m} B_n) = 0
\]

and since the sequence \( (\cup_{n \geq m} B_n)_m \) is nested, we have \( \lim_{m \to \infty} P(\cup_{n \geq m} B_n) = 0 \). Now consider the union \( \cup_{n \geq m} A_n \). We have

\[
\cup_{n \geq m} A_n \subseteq A_m \cup \cup_{n \geq m} B_n
\]

To prove this, we can show by induction on \( M \) that \( \cup_{n=M}^{M+1} A_n \subseteq A_m \cup \cup_{n \geq m} B_n \):  

- for \( M = m \), we have \( A_m \subseteq A_m \cup \cup_{n \geq m} B_n \)
- suppose the statement is true for \( M \), and consider \( \cup_{n=M}^{M+1} A_n = A_{M+1} \cup \cup_{n=M}^{M+1} A_n \). It suffices to show the inclusion for \( A_{M+1} \). Any element of \( A_{M+1} \) is either in \( A_M \), which is a subset of \( A_m \cup \cup_{n \geq m} B_n \) by induction hypothesis, or in \( A_{M+1} \setminus A_M = B_M \) which is also a subset of \( A_m \cup n \geq mB_n \). This concludes the induction.

Since the statement is true for all \( M \), the inclusion remains true for the union, which proves (1). Now from (1), we have

\[
P(\cup_{n \geq m} A_n) \leq A(A_m) + P(\cup_{n \geq m} B_n)
\]

and both terms converge to 0. Therefore \( \lim_{m \to \infty} P(\cup_{n \geq m} A_n) = 0 \), and since the sequence is nested, this is equivalent to

\[
P(\bigcap_{m=1}^{\infty} \cup_{n \geq m} A_n) = 0
\]

equivalent to

\[
P(A_n \text{ infinitely often}) = 0
\]

(4.5)

1. Let \( Z \) have standard Normal distribution. Show

\[
P(Z > z) \sim z^{-1}(2\pi)^{-1/2} \exp(-z^2/2) \text{ as } z \to \infty
\]

proof  Let \( g(z) = z^{-1}(2\pi)^{-1/2} \exp(-z^2/2) \) and

\[
F_Z(z) = P(Z > z) = (2\pi)^{-1/2} \int_{u > z} \exp(-u^2/2)du
\]

We have \( \lim_{z \to \infty} g(z) = 0 \), and \( \lim_{z \to \infty} F_Z(z) = 0 \) by the DCT (consider the sequence of functions \( f_n(u) = (2\pi)^{-1/2} \int_{u > z} \exp(-u^2/2)1_{u \geq n} \), which are dominated by \( F_Z \) and converge to 0 pointwise). Therefore By L'Hopital's rule, we have

\[
\lim_{z \to \infty} \frac{F_Z(z)}{g(z)} = \lim_{z \to \infty} \frac{F_Z'(z)}{g'(z)}
\]

where

\[
F_Z'(z) = -(2\pi)^{-1/2} \exp(-z^2/2)
\]

\[
g'(z) = (2\pi)^{-1/2} \exp(-z^2/2)(-z \frac{1}{z^2} - \frac{1}{z^2})
\]

thus

\[
\lim_{z \to \infty} \frac{F_Z(z)}{g(z)} = \lim_{z \to \infty} \frac{F_Z'(z)}{g'(z)} = \lim_{z \to \infty} \frac{1}{1 + 1/z^2} = 1
\]

which proves the result.

2. Let \( (Z_1, Z_2, \ldots) \) be independent with standard Normal distribution. Find constants \( c_n \to \infty \) such that

\[
\limsup_n Z_n/c_n = 1 \text{ a.s.}
\]
**answer** Let us find sufficient conditions on $c_n$ for the lim sup to be equal to 1.

- Consider the events $(Z_n/c_n > 1 + \epsilon)$. By the first Borel-Cantelli Lemma, if $\sum_n P(Z_n/c_n > 1 + \epsilon) < \infty$, then $P(Z_n/c_n > 1 + \epsilon \text{ infinitely often}) = 0$, i.e. with probability 1, $Z_n/c_n \leq 1 + \epsilon$ ultimately, therefore with probability 1, $\limsup_n Z_n/c_n \leq 1 + \epsilon$.

- Consider the events $(Z_n/c_n > 1 - \epsilon)$. By the second Borel-Cantelli Lemma, if $\sum_n P(Z_n/c_n > 1 - \epsilon) = \infty$, then since $Z_n/c_n$ are independent, we have $P(Z_n/c_n > 1 - \epsilon \text{ infinitely often}) = 1$, thus with probability 1, $\limsup_n Z_n/c_n \geq 1 - \epsilon$.

Therefore if $c_n$ satisfies

(a) $\sum_n P(Z_n/c_n > 1 + \epsilon) < \infty$

(b) $\sum_n P(Z_n/c_n > 1 - \epsilon) = \infty$

then $1 - \epsilon \leq \limsup_n Z_n/c_n \leq 1 + \epsilon$ for all $\epsilon > 0$, and it would follow that $\limsup_n Z_n/c_n = 1$.

By the first part, since $P(Z_n < z) \sim z^{-1}(2\pi)^{-1/2}\exp(-z^2/2)$, it suffices that

(a) $\sum_n \frac{1}{c_n} \exp(-\frac{1}{2}c_n^2(1 + \epsilon)^2) < \infty$

(b) $\sum_n \frac{1}{c_n} \exp(-\frac{1}{2}c_n^2(1 - \epsilon)^2) = \infty$

An example of sequence that satisfies these conditions is $c_n = (2\ln n)^{1/2}$, since

(a) $\frac{1}{c_n} \exp(-\frac{1}{2}c_n^2(1 + \epsilon)^2) = \frac{1}{(2\ln n)^{1/2}} \frac{1}{n^{1+\epsilon}}$ and its series converges since $(1 + \epsilon)^2 < 1$

(b) $\frac{1}{c_n} \exp(-\frac{1}{2}c_n^2(1 - \epsilon)^2) = \frac{1}{(2\ln n)^{1/2}} \frac{1}{n^{1-\epsilon}}$ and its series diverges since $(1 - \epsilon)^2 > 1$