Let $B$ be the Borel subsets of $\mathbb{R}$. For $B \in B$ define

$$
\mu(B) = \begin{cases} 
1 & \text{if } (0, \epsilon) \subset B \text{ for some } \epsilon > 0 \\
0 & \text{if not}
\end{cases}
$$

- Show that $\mu$ is not finitely additive on $B$.
- Show that $\mu$ is finitely additive but not countably additive on the field $B_0$ of finite disjoint unions of intervals $(a, b]$.

**Answer**

- For all $n \geq 1$, let $I_n = \left(\frac{1}{n+1}, \frac{1}{n}\right]$. Now consider the sets $B_0 = \bigcup_{k=1}^\infty I_{2k}$ and $B_1 = \bigcup_{k=0}^\infty I_{2k+1}$. $B_0$ and $B_1$ are countable unions of intervals, and thus Borel sets. Furthermore, $B_0$ and $B_1$ are disjoint, $\mu(B_0) = \mu(B_1) = 0$, but $B_0 \sqcup B_1 = (0, 1]$, therefore $\mu(B_0 \cup B_1) = 1$. This shows that $\mu$ is not finitely additive on $B$.

- Consider a set $B \in B_0$ of the form $B = \bigcup_{i=1}^n (a_i, b_i]$ where the union is disjoint. Then $\mu(B) = 1$ if and only if one of the intervals $(a_i, b_i]$ contains $(0, \epsilon)$ for some positive $\epsilon$, i.e. if and only if there exists $i$ such that $0 \in [a_i, b_i]$. Now if we consider two disjoint such sets $B = \bigcup_{i=1}^n (a_i, b_i]$ and $B' = \bigcup_{j=1}^m (a'_j, b'_j]$, then

  - either $\mu(B) = \mu(B') = 0$, in which case $\mu(B \sqcup B') = 0$
  - or $\mu(B) = 1$ in which case $\mu(B') = 0$ since they are disjoint, in which case $\mu(B \sqcup B') = 1$
  - or $\mu(B') = 1$ in which case $\mu(B) = 0$ since they are disjoint, in which case $\mu(B \sqcup B') = 1$

This proves that $\mu$ is finitely additive on $B_0$. However, it is not $\sigma$-additive. Indeed, consider the sets $I_n = \left(\frac{1}{n+1}, \frac{1}{n}\right]$: Each $I_n$ belongs to $B_0$, and has measure 0. However, the disjoint countable union $\bigcup_{n=1}^\infty I_n = (0, 1]$ has measure 1.

(2.2) Show that, in the definition of “a probability measure $\mu$ on a measurable space $(S, \mathcal{S})$”, we may replace “countably additive” by “finitely additive, and satisfies

$$
\mu(A_n) \downarrow 0
$$

if $A_n \downarrow \emptyset$ then $\mu(A_n) \downarrow 0$

**Proof** First note that the two following definitions of $\sigma$-additivity are equivalent for probability measures

- $A_n \uparrow A \Rightarrow \mu(A_n) \uparrow \mu(A)$
- $A_n \downarrow A \Rightarrow \mu(A_n) \downarrow \mu(A)$

(one follows from the other by taking complements). Now we show the claim.
• Suppose $\mu$ is countably additive. Then it is also finitely additive, and if $A_n \downarrow \emptyset$, then by $\sigma$-additivity, we have $\mu(A_n) \downarrow \mu(\emptyset) = 0$

• Now suppose that $\mu$ is finitely additive and for any nested sequence $A_n \downarrow \emptyset$, $\mu(A_n) \downarrow 0$. Now consider any nested sequence $B_n \downarrow B$, where $B$ is the intersection $B = \cap B_n$. For all $n$, define $A_n = B_n \setminus B$. Then we have $A_n$ is a nested sequence, and $A_n \downarrow \emptyset$ (indeed, $\cap A_n = \cap (B_n \cap B^c) = (\cap B_n) \cap B^c = B \cap B^c = \emptyset$). Therefore $\mu(A_n) \downarrow 0$. But since $B \subset B_n$, we have $B_n = A_n \cup B$ where the union is disjoint. Thus by finite additivity, $\mu(B_n) = \mu(A_n) + \mu(B)$. Finally, since $\mu(A_n) \downarrow 0$, we have $\mu(B_n) \downarrow \mu(B)$.

(2.3) Give an example of a measurable space $(S, \mu)$, a collection $\mathcal{A}$ and probability measures $\mu$ and $\nu$ such that

• $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$
• $S = \sigma(\mathcal{A})$
• $\mu \neq \nu$

**answer** Consider $S = \{1, 2, 3, 4\}$, $\mathcal{S} = P(S)$, $\mathcal{A} = \{\{1, 2\}, \{2, 3\}\}$. Note that we have $\sigma(\mathcal{A}) = S$. Now define $\mu$ and $\nu$ by

• $\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = 1/3$ and $\mu(\{4\}) = 0$ (this completely determines $\mu$ by finite additivity)
• $\nu(\{1\}) = \nu(\{3\}) = 0, \nu(\{2\}) = 2/3$ and $\nu(\{4\}) = 1/3$ (this completely determines $\nu$ by finite additivity).

Then we have $\mu(\{1, 2\}) = \nu(\{1, 2\}) = 2/3$ and $\mu(\{2, 3\}) = \nu(\{2, 3\}) = 2/3$, i.e. $\mu$ and $\nu$ agree on $\mathcal{A}$, but $\mu$ and $\nu$ are different.

(2.4) Let $\mu$ be a probability measure on $(S, \mathcal{S})$, where $\mathcal{S} = \sigma(\mathcal{F})$ for a field $\mathcal{F}$. Show that for each $B \in \mathcal{S}$ and $\epsilon > 0$, there exists $A \in \mathcal{F}$ such that $\mu(B \Delta A) < \epsilon$

**proof** Since $\mathcal{F}$ is a field, it is in particular closed under intersection, thus it forms a $\pi$-class of subsets of $S$. Now let $\mathcal{B}$ be the collection of subsets

$$
\mathcal{B} = \{B \subset S : \forall \epsilon > 0, \exists A \in \mathcal{F} : \mu(B \Delta A) < \epsilon \}
$$

it suffices to show that $\mathcal{B}$ is a $\lambda$-class, then by Dynkin’s Lemma, we have $\sigma(\mathcal{F}) \subset \mathcal{B}$, i.e. $\mathcal{S} \subset \mathcal{B}$, which proves the claim.

Now let us show that $\mathcal{B}$ is a $\lambda$-class that contains $\mathcal{F}$:

• we have $S \in \mathcal{B}$: indeed, for all $\epsilon > 0$, we have $\mu(S \Delta S) = \mu(\emptyset) = 0 < \epsilon$, and $S \in \mathcal{F}$ (since $\mathcal{F}$ is a field)

• let $B_1, B_2 \in \mathcal{B}$. We seek to show that $B = B_1 \setminus B_2 \in \mathcal{B}$. Fix $\epsilon > 0$. There exist $A_1, A_2$ such that $\mu(B_1 \Delta A_1) < \epsilon/2$ and $\mu(B_2 \Delta A_2) < \epsilon/2$. Now consider $A = A_1 \setminus A_2$. We have

$$
B \Delta A = (B \cap A^c) \cup (B^c \cap A)
$$

the first term is $B \cap A^c = (B_1 \cap B_2^c) \cap (A_1^c \cup A_2) = (B_1 \cap B_2^c \cap A_1^c) \cup (B_1 \cap B_2^c \cap A_2) \subseteq (B_1 \cap A_1^c) \cup (B_2^c \cap A_2) \subseteq (B_1 \Delta A_1) \cup (B_2 \Delta A_2)$. Similarly, the second term is also a subset of $(B_1 \Delta A_1) \cup (B_2 \Delta A_2)$ (by symmetry in $A, B$). Therefore

$$
B \Delta A \subseteq (B_1 \Delta A_1) \cup (B_2 \Delta A_2)
$$

and its measure is $\leq \mu((B_1 \Delta A_1) \cup (B_2 \Delta A_2)) \leq \mu(B_1 \Delta A_1) + \mu(B_2 \Delta A_2) < \epsilon$. 

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• Let $B_n$ be a nested sequence of elements of $\mathcal{B}$, such that $B_n \uparrow B$. We seek to show that $B \in \mathcal{B}$. Fix $\epsilon > 0$. Since $B_n \uparrow B$, then $\mu(B_n) \uparrow \mu(B)$. Thus there exists $N$ such that $\mu(B_N) > \mu(B) - \epsilon/2$. Since $B_N \in \mathcal{B}$, there exists $A \in \mathcal{F}$ such that $\mu(B_N \Delta A) \leq \epsilon/2$. Now consider $B \Delta A$. We have can write

$$B \Delta A = (B \cap A) \cup (B \cap A^c)$$

$$= (B_N \cap Z^c \cap A) \cup ((B_N \cup Z) \cap A^c)$$

$$\subseteq (B_N \cap A) \cup (B_N \cap A^c) \cup (Z \cap A^c)$$

$$\subseteq (B_N \Delta A) \cup Z$$

therefore

$$\mu(B \Delta A) \leq \mu(B_N \Delta A) + \mu(Z) < \epsilon/2 + \epsilon/2$$

and it follows that $B \in \mathcal{B}$

Finally, $\mathcal{B}$ contains $\mathcal{F}$ since for all $B \in \mathcal{F}$ and for all $\epsilon > 0$, $\mu(B \Delta B) = \mu(\emptyset) = 0 < \epsilon$. This concludes the proof.

(2.5) Let $g : [0, 1] \rightarrow \mathbb{R}$ be integrable w.r.t. Lebesgue measure. Let $\epsilon > 0$. Show that there exists a continuous function $f : [0, 1] \rightarrow \mathbb{R}$ such that $\int |f(x) - g(x)|dx \leq \epsilon$.

**proof** We first show that there exists a simple function $s : [0, 1] \rightarrow \mathbb{R}$ such that $\int |s(x) - g(x)| \leq \epsilon/2$. This follows from the definition of the integral of an integrable function

$$\int gd\mu = \int g^+d\mu - \int g^-d\mu$$

where $\int g^+d\mu = \sup \int_{s\text{ simple }, s \leq g^+} sd\mu$ (and similarly for $g^-$, both functions being non-negative). Therefore there exist simple functions $s_1$ and $s_2$ such that

$$s_1 \leq g^+$$

$$s_2 \leq g^-$$

$$\int s_1 \geq \int g^+d\mu - \epsilon/4$$

$$\int s_2 \geq \int g^-d\mu - \epsilon/4$$

define a simple function $s = s_1 - s_2$. Then we have

$$\int |s - g|d\mu = \int |s_1 - s_2 - g^+ + g^-|$$

$$\leq \int |s_1 - g^+|d\mu + \int |s_2 - g^-|d\mu$$

by the triangle inequality

$$\leq \epsilon/4 + \epsilon/4$$

by definition of $s_1$, $s_2$

Now it suffices to show that any simple function $s$ defined on $[0, 1]$ can be approximated by a continuous function on $[0, 1]$, in the following sense: there exists $f : [0, 1] \rightarrow \mathbb{R}$ continuous, such that $\int |s - f| \leq \epsilon/2$. We can write $s$ as the finite sum of scaled indicator functions

$$s = \sum_{i=1}^{n} c_i 1_{A_i}$$
where for all $i$, $A_i \subseteq [0,1]$ is a Lebesgue measurable set, and the $A_i$’s are disjoint. By regularity of the Lebesgue measure, there exist a compact set $K_i$ and an open set $O_i$ such that

$$K_i \subseteq A_i \subseteq O_i,$$

$$\mu(A_i \setminus K_i) \leq \frac{\epsilon}{4|c_i|n},$$

$$\mu(O_i \setminus A_i) \leq \frac{\epsilon}{4|c_i|n}$$

in particular, we have $\mu(O_i \cap K_i^c) \leq \frac{\epsilon}{2|c_i|n}$. Now consider the closed sets $O_i^c$ and $K_i$. By Urysohn’s Lemma, there exists a continuous function $f_i : [0,1] \to \mathbb{R}$ such that $f_i$ is identically 0 on $O_i^c$, and identically 1 on $K_i$, therefore

- $|f_i(x) - 1_{A_i}(x)| = 0$ for all $x \in O_i^c \cup K_i$
- $|f_i(x) - 1_{A_i}(x)| \leq 1$ for all $x \in (O_i^c \cup K_i)^c = O_i \cap K_i^c$
- $\int |f_i - 1_{A_i}| \mu \leq \mu(O_i \cap K_i^c) \leq \frac{\epsilon}{2|c_i|n}$

Now consider the function $f = \sum_{i=1}^n c_i f_i$. We have $f$ is continuous as the finite sum of continuous functions, and

$$\int |s - f| \mu = \int |\sum_{i=1}^n c_i (1_{A_i} - f_i)| \mu$$

$$\leq \sum_{i=1}^n |c_i| \int |f_i - 1_{A_i}|$$

$$\leq \sum_{i=1}^n |c_i| \frac{\epsilon}{2|c_i|n}$$

$$= \frac{\epsilon}{2}$$

This concludes the proof.