(1.1) Let $F_n$ be collections of subsets of a set $S$. Suppose that each $F_n$ is a field, and $F_n \subset F_{n+1}$ for all $n$. Define $F = \cup_n F_n$. Show that $F$ is a field. Give an example to show that $F$ need not be a $\sigma$-field.

**answer** $F$ is nonempty since $F_1$ is non-empty. It suffices to check that $F$ is closed under union and complement.

- Let $A \in F$. Then $\exists n$ such that $A \in F_n$, and since $F_n$ is a field, $A^c \in F_n \subset F$.
- Let $A_1, A_2 \in F$. Then there exist $n_1, n_2$ such that $A_1 \in F_{n_1}$, $A_2 \in F_{n_2}$; assume without loss of generality that $n_1 \leq n_2$. Then since $F_{n_1} \subset F_{n_2}$, $A_1$ and $A_2$ are both elements of $F_{n_2}$, and since $F_{n_2}$ is a field, $A \cup B \in F_{n_2} \subset F$.

Therefore $F$ is a field. $F$ is not necessarily a $\sigma$-field. Indeed, consider the case where $S = \mathbb{R}$ and $F_n = F(A)$ for all $n$, the field generated by $A$, the set of open intervals of the form $(-\infty, a)$, $a \in \mathbb{R}$. Then the union $F$ is a field, but it is not a $\sigma$-field, since the $\sigma$-field generated by $A$ contains for example countable unions of disjoint open intervals, and these sets are not in $F$.

(1.2) Given a nonempty collection $A$ of sets, we defined $F(A)$ as the intersection of all fields containing $A$. Show that $F(A)$ is the class of sets of the form

$$\cup_{i=1}^{m} \cap_{j=1}^{n_i} A_{i,j}$$

where for each $i$ and $j$, either $A_{i,j} \in A$ or $A_{i,j}^c \in A$, and the sets $\cap_{j=1}^{n_i} A_{i,j}$, $i \in \{1, \ldots, m\}$ are pairwise disjoints.

**proof** Let $B$ be this class of sets. $B$ is a subset of any field that contains $A$, therefore $B \subseteq F(A)$. It remains to show that $B$ is a field.

- First, $B$ contains the empty set: take any element $A \in A$, then set $A_{1,1} = A$, $A_{1,2} = A^c$. Then by definition, $B$ contains $A_{1,1} \cap A_{1,2} = \emptyset$ (taking $n = 1$ in the expression (1)).
- $B$ is closed under intersection: let $B = \cup_i \cap_j A_{i,j}$ and $B' = \cup_k \cap_l A'_{k,l}$ be two elements of $B$, and call $B_i = \cap_j A_{i,j}$ and $B'_k = \cap_l A'_{k,l}$. By assumption, the $B_i$’s are pairwise disjoint, and so are the $B'_k$’s. Then we have

$$B \cap B' = \cup_{i,k} (B_i \cap B'_k)$$

where the sets $B_i \cap B'_k$ are pairwise disjoint, and can be further expanded into an intersection of the form

$$B_i \cap B'_k = (\cap_j A_{i,j}) \cap (\cap_l A'_{k,l})$$

where by assumption, each $A_{i,j}$ and $A'_{k,l}$ is either an element of $A$ or has its complement in $A$.
- $B$ is closed under complement: let $B = \cup_i \cap_j A_{i,j}$. Then

$$B^c = \cap_i \cup_j A_{i,j}^c$$

for each $i$, $\cup_j A_{i,j}^c$ is in $B$ since it is of the form 1 (taking $n_i$ to be 1). Finally, since $B$ is closed under intersection, the intersection $\cap_i \cup_j A_{i,j}^c$ is in $B$. This concludes the proof.
(1.3) Suppose $B \in \sigma(A)$ for some collection $A$ of subsets. Show there exists a countable subcollection $A_B \subset A$ such that $B \in \sigma(A_B)$.

**proof** Consider the union

$$B = \bigcup_{C \subset A, C \in \text{countable}} \sigma(C)$$

the claim is that $\sigma(A) \subset B$. To prove this, it suffices to show that $B$ is a $\sigma$-field that contains all elements of $A$.

- Let $A \in A$. We have $\{A\}$ is a countable (finite) subset of $A$, thus $\sigma(\{A\}) \subset B$, therefore $A \in \sigma(\{A\}) \subset B$ which proves that all elements of $A$ are in $B$.
- $B$ is closed under countable union: consider a sequence $(B_n)$ of elements of $B$. By definition, for each $n$, there exists $C_n$ countable subset of $A$ such that $B_n \in \sigma(C)$. Now let $C = \bigcup_n C_n$, which is also a countable subset of $A$ (a countable union of countable sets is countable) thus $\sigma(C) \subset B$. And since $C$ is a $\sigma$-field and contains each $B_n$, it also contains the countable union $\cup_n B_n$. This proves that $\cup_n B_n \in B$.
- $B$ is closed under complement: let $B \in B$. Then there exists $C$ countable subset of $A$ such that $B \in \sigma(C)$. By definition of a $\sigma$-field, $\sigma(C)$ also contains the complement $B^c$, therefore $B^c \in \sigma(C) \subset B$.

This completes the proof that $B$ is a $\sigma$-field that contains all elements of $A$.

(1.4) Show that the Borel $\sigma$-field on $\mathbb{R}^d$ is the smallest $\sigma$-field that makes all continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ measurable.

**proof** Let $B$ be the Borel $\sigma$-field on $\mathbb{R}^d$ (smallest $\sigma$-field that contains open sets), and let $C$ be the smallest $\sigma$-field that makes all continuous functions measurable. The claim is that $B = C$.

First, let $f$ be a continuous function. Then $f$ is measurable for $B$ (it suffices to show that $f^{-1}((-\infty, a])$ is measurable for any $a \in \mathbb{R}$, but since $f$ is continuous the inverse image of open sets is open, thus in $B$). Therefore $B$ is a $\sigma$-field that makes all continuous functions measurable, therefore $C \subseteq B$.

To show the reverse inclusion, it suffices to show that $C$ contains closed sets. Let $C$ be a closed set in $\mathbb{R}^d$, and consider the function

$$f_C : \mathbb{R}^d \to \mathbb{R} \quad x \mapsto d(x, C)$$

where the distance is defined as $d(x, C) = \inf_{y \in C} \|x - y\|_2$ for example. Here the inf is in fact attained, but this property is not needed. $f_C$ is continuous, since $|f(x_1) - f(x_2)| \leq \|x_1 - x_2\|_2$ for all $x_1, x_2 \in C$. Therefore $C$ contains $f^{-1}([0])$, which is exactly $C$. Therefore $C$ contains all closed sets, and since it is a $\sigma$-field, we must have $B \subseteq C$.

(1.5) A function $f : \mathbb{R}^d \to \mathbb{R}$ is lower semicontinuous (l.s.c.) if $\liminf_{y \to x} f(y) \leq f(x)$ for all $x$. A function is upper semicontinuous (u.s.c.) if $\limsup_{y \to x} f(y) \leq f(x)$ for all $x$. Show that if $f$ is l.s.c. or u.s.c., then $f$ is measurable.

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1 one way to show this is by definition of the distance: for all $\epsilon > 0$, there exists $y_1$ such that $f_C(x_1) \geq \|x_1 - y_1\| - \epsilon$. Then

$$f_C(x_2) \leq \|x_2 - y_1\|$$

$$\leq \|x_2 - x_1\| + \|x_1 - y_1\|$$

by the triangle inequality

$$\leq \|x_2 - x_1\| + f_C(x_1) + \epsilon$$

since this holds for any $\epsilon > 0$, we have $f_C(x_2) \leq \|x_2 - x_1\| + f_C(x_1)$. Then by symmetry in $x_1, x_2$, it follows that $|f_C(x_2) - f_C(x_1)| \leq \|x_2 - x_1\|_2$. 

\[ \text{2} \]
Proof Let $f$ be lower semicontinuous. To show that $f$ is measurable, it suffices to show that $f^{-1}((\infty, a))$ is a measurable set for all $a \in \mathbb{R}$. Let $a \in \mathbb{R}$, and call $F_a = f^{-1}((\infty, a))$. The claim is that $F_a$ is an open set. Indeed, let $x \in F_a$. Then $f(x) < a$. Now since $f$ is l.s.c., we have $\limsup_{y \to x} f(y) \leq f(x)$, thus for all $\epsilon > 0$, there exists $\delta > 0$ such that $\inf_{y \in B(x, \delta)} f(y) < \limsup_{y \to x} f(y) + \epsilon \leq f(x) + \epsilon$. In particular for $\epsilon = a - f(x)$, there exists $\delta > 0$ such that $\sup_{y \in B(x, \delta)} f(y) < f(x) + \epsilon = a$, i.e. $B(x, \delta) \subseteq F_a$. This proves that every point in $F_a$ is in the interior of $F_a$, i.e. that $F_a$ is open, hence $F_a$ is measurable, which completes the proof.

For the case of upper semi-continuous functions, consider the function $-f$, which is lower semicontinuous. By the previous argument, $-f$ is measurable, and so is $f$ (product of two measurable functions is measurable).