STAT 205A - Problem Set 04

Walid Krichene (23265217)

October 1, 2013

(4.1. Monte Carlo Integration) Let $f : [0,1] \to \mathbb{R}$ be such that $\int_{[0,1]} f^2(x)dx < \infty$. Let U_i be i.i.d. Uniform$(0,1)$. Let

$$D_n = n^{-1} \sum_{i=1}^{n} f(U_i) - \int_0^1 f(x)dx$$

1. Use Chebyshev's inequality to bound $P(|D_n| > \epsilon)$

   answer The density function of U is simply the indicator function of $[0,1]$. Therefore the expectation of D_n is given by

$$\mathrm{E}\, D_n = n^{-1} \sum_{i=1}^{n} \int_{u=0}^{1} f(u)du - \int_{x=0}^{1} f(x)dx = 0$$

   Now by Chebyshev's inequality, we have

$$P(|D_n| > \epsilon) \leq \frac{1}{\epsilon^2} \operatorname{var}(D_n)$$

   writing $\int_0^1 f(x)dx = \mathrm{E}\, f(I)$, we have

$$\begin{aligned}
\operatorname{var}(D_n) &= \mathrm{E}\left(\frac{1}{n}\sum_{i=1}^{n} f(U_i) - \mathrm{E}\, f(U)\right)^2 && \text{since } \mathrm{E}\, D_n = 0 \\
&= \frac{1}{n^2}\, \mathrm{E}\left(\sum_{i=1}^{n} f(U_i)\right)^2 + (\mathrm{E}\, f(U))^2 - \frac{2}{n}\, \mathrm{E}\, f(U) \sum_{i=1}^{n} \mathrm{E}\, f(U_i) \\
&= \frac{1}{n^2}\left(\sum_{i=1}^{n} \mathrm{E}\, f^2(U_i) + 2 \sum_{1\leq i<j\leq n} \mathrm{E}(f(U_i)f(U_j))\right) + (\mathrm{E}\, f(U))^2 - 2\,(\mathrm{E}\, f(U))^2 \\
&= \frac{1}{n^2}\left(\sum_{i=1}^{n} \mathrm{E}\, f^2(U_i) + 2 \sum_{1\leq i<j\leq n} \mathrm{E}\, f(U_i)\, \mathrm{E}\, f(U_j))\right) - (\mathrm{E}\, f(U))^2 && \text{by independence of } f(U_i)\text{'s} \\
&= \frac{1}{n}\, \mathrm{E}\, f^2(U) + \frac{2n(n-1)}{n^2}\, (\mathrm{E}\, f(U))^2 - (\mathrm{E}\, f(U))^2 \\
&= \frac{1}{n}\, \mathrm{E}\, f^2(U) + \frac{n-2}{n}\, (\mathrm{E}\, f(U))^2
\end{aligned}$$

   which written in integral form, is

$$\operatorname{var}(D_n) = \frac{1}{n}\int_{u=0}^{1} f^2(u)du + \frac{n-2}{n}\left(\int_0^1 f(u)du\right)^2$$

2. Show this bound remains true if the U_i are only *pairwise* independent.

proof Pairwise independence is sufficient since in the above proof, it suffices that $\mathrm{E}\, f(U_i)f(U_j) = \mathrm{E}\, f(U_i)\,\mathrm{E}\, f(U_j)$ for all i, j, which is true when U_i's are pairwise independent.

(4.2) Let $X \geq 0$ and $Y \geq 0$ be independent r.v.'s with densities f_X and f_Y. Calculate the densities of XY and X/Y

answer

- Let f_{XY} be the density of XY. Then for all $c \in \mathbb{R}$,

$$P(XY \leq c) = \int_{u=0}^{c} f_{XY}(u)du$$

but we can also write

$$\begin{aligned}
P(XY \leq c) &= \iint_{x\in\mathbb{R}_+, y\in\mathbb{R}_+} 1_{xy\leq c} f_X(x) f_Y(y) dxdy \\
&= \int_{x\in\mathbb{R}_+\setminus\{0\}} \int_{y\in\mathbb{R}_+} 1_{xy\leq c} f_X(x) f_Y(y) dxdy && \text{since } \{0\} \text{ has measure } 0 \\
&= \int_{x\in\mathbb{R}_+\setminus\{0\}} \left(\int_{y=0}^{c/x} f_Y(y)dy \right) f_X(x)dx \\
&= \int_{x\in\mathbb{R}_+\setminus\{0\}} \left(\int_{u=c}^{0} \frac{1}{x} f_Y(u/x)du \right) f_X(x)dx && \text{using the change of variable } u = xy \\
&= \int_{u=0}^{c} \left(\int_{x\in\mathbb{R}_+\setminus\{0\}} \frac{1}{x} f_Y(u/x) f_X(x)dx \right) du && \text{by Fubini}
\end{aligned}$$

therefore we have for all c,

$$\int_{u=0}^{c} f_{XY}(u)du = \int_{u=0}^{c} \left(\int_{x\in\mathbb{R}_+\setminus\{0\}} \frac{1}{x} f_Y(u/x) f_X(x)dx \right) du$$

and taking derivative with respect to c on both sides, we conclude that

$$f_{XY}(u) = \left(\int_{x\in\mathbb{R}_+\setminus\{0\}} \frac{1}{x} f_Y(u/x) f_X(x)dx \right)$$

- Let $f_{X/Y}$ be the density of X/Y. Then for all $c \in \mathbb{R}$,

$$P(X/Y \leq c) = \int_{u=0}^{c} f_{X/Y}(u)du$$

but we can also write

$$\begin{aligned}
P(X/Y \leq c) &= \iint_{x\in\mathbb{R}_+, y\in\mathbb{R}_+\setminus\{0\}} 1_{x/y\leq c} f_X(x) f_Y(y) dxdy && \text{since } X \text{ and } Y \text{ are independent} \\
&= \int_{y\in\mathbb{R}^+\setminus\{0\}} \left(\int_{x=0}^{yc} f_X(x)dx \right) f_Y(y)dy && \text{B Fubini} \\
&= \int_{y\in\mathbb{R}_+\setminus\{0\}} \left(\int_{u=0}^{c} y f_X(uy)du \right) f_Y(y)dy && \text{using the change of variable } u = x/y \\
&= \int_{u=0}^{c} \left(\int_{y\in\mathbb{R}_+\setminus\{0\}} y f_X(uy) f_Y(y)dy \right) du && \text{by Fubini}
\end{aligned}$$

therefore we have for all c,

$$\int_{u=0}^{c} f_{X/Y}(u)du = \int_{u=0}^{c} \left(\int_{y\in\mathbb{R}_+\setminus\{0\}} y f_X(uy) f_Y(y) dy \right) du$$

and taking derivative with respect to c on both sides, we conclude that

$$f_{X/Y}(u) = \left(\int_{y\in\mathbb{R}_+\setminus\{0\}} y f_X(uy) f_Y(y) dy \right)$$

(4.3) Let (X_i) be r.v.'s with $\mathrm{E}\, X_i = 0$ and $\mathrm{E}\, X_i X_j \le r(j-i)$ for $1 \le i \le j < \infty$, where $r(n)$ is a deterministic sequence with $r(n) \to 0$ as $n \to \infty$. Prove that $n^{-1} \sum_{i=1}^{n} X_i \to 0$ in Probability.

proof Let $S_n = \sum_{i=1}^{n} X_i$. By Chebyshev's inequality, we have

$$\begin{aligned} P(\frac{|S_n|}{n} > \epsilon) &\le \frac{1}{n^2\epsilon^2} \mathrm{E}[S_n^2] \\ &\le \frac{1}{n^2\epsilon^2} \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{E}[X_i X_j] && \text{by linearity of the expectation} \\ &\le \frac{1}{n^2\epsilon^2} \sum_{i=1}^{n}\sum_{j=1}^{n} r(|j-i|) \end{aligned}$$

Now for any fixed i,

$$\sum_{j=1}^{n} r(|j-i|) = \sum_{j=1}^{i} r(i-j) + \sum_{j=i+1}^{n} r(j-i) = \sum_{k=0}^{i-1} r(k) + \sum_{l=1}^{n-i} r(l) \le 2\sum_{k=1}^{n} |r(k)|$$

Therefore

$$\begin{aligned} P(\frac{|S_n|}{n} > \epsilon) &\le \frac{1}{n^2\epsilon^2} 2n \sum_{k=1}^{n} |r(k)| \\ &= \frac{2}{\epsilon^2} \frac{\sum_{k=1}^{n} |r(k)|}{n} \end{aligned}$$

and since $(r(n))_n$ converges to 0, so does $(|r(n)|)_n$, and so does the sequence of Cesàro means $(\frac{\sum_{k=1}^{n} |r_k||}{n})_n$ (for any $\delta > 0$, there exists N such that for all $n > N$, $|r_n| < \delta$. Then we can write $\frac{1}{n}\sum_{k=1}^{n} |r_k| = \frac{1}{n}\sum_{k=1}^{N} |r_k| + \frac{n-N}{n}\delta \le \frac{\sum_{k=1}^{N} |r_k|}{n} + \delta \le 2\delta$ for n large enough).

Therefore $P(\frac{|S_n|}{n} > \epsilon)$ converges to 0 as $n \to \infty$

(4.4) Suppose events A_n satisfy $P(A_n) \to 0$ and

$$\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty$$

Prove that

$$P(A_n \text{ infinitely often}) = 0$$

proof Let us write $B_n = A_{n+1} \setminus A_n = A_{n+1} \cap A_n^c$.

By the first Borel-Cantelli Lemma, applied to the events B_n, we have $P(B_n \text{ infinitely often}) = 0$, i.e.

$$P(\cap_{m=1}^{\infty} \cup_{n\geq m} B_n) = 0$$

and since the sequence $(\cup_{n\geq m} B_n)_m$ is nested, we have $\lim_{m\to\infty} P(\cup_{n\geq m} B_n) = 0$. Now consider the union $\cup_{n\geq m} A_n$. We have

$$\cup_{n\geq m} A_n \subseteq A_m \cup \cup_{n\geq m} B_n \tag{1}$$

To prove this, we can show by induction on M that $\cup_{n=m}^{M} A_n \subseteq A_m \cup \cup_{n\geq m} B_n$:

- for $M = m$, we have $A_m \subseteq A_m \cup \cup_{n\geq m} B_n$
- suppose the statement is true for M, and consider $\cup_{n=m}^{M+1} A_m = A_{M+1} \cup \cup_{m=1}^{M} A_m$. It suffices to show the inclusion for A_{M+1}. Any element of A_{M+1} is either in A_M, which is a subset of $A_m \cup \cup_{n\geq m} B_n$ by induction hypothesis, or in $A_{M+1} \setminus A_M = B_M$ which is also a subset of $A_m \cup n \geq m B_n$. This concludes the induction.

Since the statement is true for all M, the inclusion remains true for the union, which proves (1). Now from (1), we have

$$P(\cup_{n\geq m} A_n) \leq A(A_m) + P(\cup_{n\geq m} B_n)$$

and both terms converge to 0. Therefore $\lim_{m\to\infty} P(\cup_{n\geq m} A_n) = 0$, and since the sequence is nested, this is equivalent to

$$P(\cap_{m=1}^{\infty} \cup_{n\geq m} A_n) = 0$$

i.e.

$$P(A_n \text{ infinitely often}) = 0$$

(4.5)

1. Let Z have standard Normal distribution. Show

   $$P(Z > z) \sim z^{-1}(2\pi)^{-1/2} \exp(-z^2/2) \text{ as } z \to \infty$$

   proof Let $g(z) = z^{-1}(2\pi)^{-1/2} \exp(-z^2/2)$ and

   $$F_Z(z) = P(Z > z) = (2\pi)^{-1/2} \int_{u>z} \exp(-u^2/2) du$$

   We have $\lim_{z\to\infty} g(z) = 0$, and $\lim_{z\to\infty} F_Z(z) = 0$ by the *DCT* (consider the sequence of functions $f_n(u) = (2\pi)^{-1/2} \int_{u>z} \exp(-u^2/2) 1_{u\geq n}$, which are dominated by F_Z and converge to 0 pointwise). Therefore By L'Hopital's rule, we have

   $$\lim_{z\to\infty} \frac{F_Z(z)}{g(z)} = \lim_{z\to\infty} \frac{F_Z'(z)}{g'(z)}$$

   where

   $$F_Z'(z) = -(2\pi)^{-1/2} \exp(-z^2/2)$$
   $$g'(z) = (2\pi)^{-1/2} \exp(-z^2/2)(-z\frac{1}{z} - \frac{1}{z^2})$$

   thus

   $$\lim_{z\to\infty} \frac{F_Z(z)}{g(z)} = \lim_{z\to\infty} \frac{F_Z'(z)}{g'(z)} = \lim_{z\to\infty} \frac{1}{1 + 1/z^2} = 1$$

   which proves the result.

2. Let $(Z_1, Z_2, \ldots)$ be independent with standard Normal distribution. Find constants $c_n \to \infty$ such that

   $$\limsup_n Z_n/c_n = 1 \text{ a.s.}$$

answer Let us find sufficient conditions on c_n for the lim sup to be equal to 1.

- Consider the events $(Z_n/c_n > 1+\epsilon)$. By the first Borel-Cantelli Lemma, if $\sum_n P(Z_n/c_n > 1+\epsilon) < \infty$, then $P(Z_n/c_n > 1+\epsilon$ infinitely often$) = 0$, i.e. with probability 1, $Z_n/c_n \leq 1+\epsilon$ ultimately, therefore with probability 1, $\limsup_n Z_n/c_n \leq 1+\epsilon$.
- Consider the events $(Z_n/c_n > 1-\epsilon)$. By the second Borel-Cantelli Lemma, if $\sum_n P(Z_n/c_n > 1-\epsilon) = \infty$, then since Z_n/c_n are independent, we have $P(Z_n/c_n > 1-\epsilon$ infinitely often$) = 1$, thus with probability 1, $\limsup_n Z_n/c_n \geq 1-\epsilon$.

Therefore if c_n satisfies

(a) $\sum_n P(Z_n/c_n > 1+\epsilon) < \infty$

(b) $\sum_n P(Z_n/c_n > 1-\epsilon) = \infty$

then $1-\epsilon \leq \limsup_n Z_n/c_n \leq 1+\epsilon$ for all $\epsilon > 0$, and it would follow that $\limsup_n Z_n/c_n = 1$.

By the first part, since $P(Z_n < z) \sim z^{-1}(2\pi)^{-1/2}\exp(-z^2/2)$, it suffices that

(a) $\sum_n \frac{1}{c_n}\exp(-\frac{1}{2}c_n^2(1+\epsilon)^2) < \infty$

(b) $\sum_n \frac{1}{c_n}\exp(-\frac{1}{2}c_n^2(1-\epsilon)^2) = \infty$

An example of sequence that satisfies these conditions is $c_n = (2\ln n)^{1/2}$, since

(a) $\frac{1}{c_n}\exp(-\frac{1}{2}c_n^2(1+\epsilon)^2) = \frac{1}{(2\ln n)^{1/2}}\frac{1}{n^{(1+\epsilon)^2}}$ and its series converges since $(1+\epsilon)^2 < 1$

(b) $\frac{1}{c_n}\exp(-\frac{1}{2}c_n^2(1-\epsilon)^2) = \frac{1}{(2\ln n)^{1/2}}\frac{1}{n^{(1-\epsilon)^2}}$ and its series diverges since $(1-\epsilon)^2 > 1$