# STAT 205A - Problem Set 03

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**(3.1)**

1. If $X_n \geq 0$, $X_n \downarrow X$ a.s. and $EX_N < \infty$ for some $N$, then $\mathrm{E}\, X_n \to \mathrm{E}\, X$

   ***proof*** Consider the variable $Y_n = X_N - X_n$, for $n \geq N$. Since $X_n$ is decreasing, $(Y_n)_{n \geq N}$ is non-negative increasing, and its limit is $X_N - X$. Therefore by the MCT, we have

   $$\lim_{n\to\infty} \mathrm{E}(Y_n) = \mathrm{E}\, X_N - \mathrm{E}\, X$$

   and using linearity of the expectation,

   $$\lim_{n\to\infty} \mathrm{E}(X_N) - \mathrm{E}(X_n) = \mathrm{E}\, X_N - \mathrm{E}\, X$$

   finally, since $\mathrm{E}\, X_N < \infty$, we have

   $$\lim_{n\to\infty} \mathrm{E}(X_n) = \mathrm{E}\, X$$

2. if $\mathrm{E}\,|X| < \infty$ then $E|X|1_{(|X|>n)} \to 0$ as $n \to \infty$

   ***proof*** Let $A = \{\omega : |X(\omega)| < \infty\}$. Since $\mathrm{E}\,|X| < \infty$, we have $\mu(A^c) = 0$, and we can restrict the integral to $A^c$:

   $$\mathrm{E}\,|X| = \int_\Omega |X(\omega)| 1_A(\omega) d\mu(\omega)$$

   now consider the sequence of functions

   $$f_n(\omega) = |X(\omega)| 1_{|X|>n}(\omega)$$

   Then

   - $f_n$ is non-negative, measurable
   - for all $\omega$, $f_n(\omega) \uparrow |X(\omega)| 1_A$ as $n \to \infty$

   Thus by the MCT, we have $\lim_{n\to\infty} \mathrm{E}\, f_n(\omega) = \mathrm{E}\, X$, which is the desired result.

3. if $\mathrm{E}\,|X_1| < \infty$ and $X_n \uparrow X$ a.s., then either $\mathrm{E}\, X_n \uparrow \mathrm{E}\, X < \infty$ or else $\mathrm{E}\, X_n \uparrow \infty$ and $\mathrm{E}\,|X| = \infty$

***proof*** First, since $X_1$ is integrable, we have $\mathrm{E}\,X_1^+ < \infty$ and $\mathrm{E}\,X_1^- < \infty$. Now since $X_n \uparrow X$, we have $X_n^+ \uparrow X^+$ and $X_n^- \downarrow X^-$, where $X_n^+$ and $X_n^-$ are non-negative. We have

- Since $\mathrm{E}\,X_1^- < \infty$, all the terms of the sequence are finite, and we have (by the MCT) $\mathrm{E}\,X_n^- \downarrow \mathrm{E}\,X^-$, which is finite.
- By the MCT, $\mathrm{E}\,X_n^+ \uparrow \mathrm{E}\,X^+$. Two cases are possible
  (a) either the limit $\mathrm{E}\,X^+ = \infty$, in which case $\mathrm{E}\,X_n = \mathrm{E}\,X_n^+ - \mathrm{E}\,X_n^-$ is the sum of a sequence that diverges to $\infty$ and a bounded sequence, therefore it diverges, and $\mathrm{E}\,X = \mathrm{E}\,X^+ + \mathrm{E}\,X^- = \infty$.
  (b) either the limit $\mathrm{E}\,X^+ < \infty$, in which case $\mathrm{E}\,X_n = \mathrm{E}\,X_n^+ - \mathrm{E}\,X_n^-$ is the sum of two converging sequences, thus it converges to the sum of limits, i.e. $\mathrm{E}\,X_n \to \mathrm{E}\,X < \infty$.

4. if $X$ takes values in the non-negative integers, then

$$\mathrm{E}\,X = \sum_{n=1}^{\infty} P(X \geq n)$$

***proof*** By definition, the expectation of $X$ is given by

$$\mathrm{E}\,X = \sum_{i=1}^{\infty} iP(X = i)$$

which is the sum of the series with terms $f_i = iP(X = i)$. Now consider the sequence of series $f^{(m)}$, $m \geq 1$, where each $f^{(m)}$ is given by

$$f_i^{(m)} = \sum_{n=1}^{m} P(X = i)1_{n \leq i}$$

then we have

- $f^{(m)}$ is non-negative, measurable for all $m$
- for all $i$, $\lim_{m\to\infty} f^(m)_i = \sum_{n\leq i} P(X = i) = iP(X = i) = f_i$

Therefore by the MCT, we have

$$\sum_{i=1}^{\infty} f_i = \lim_{m\to\infty} \sum_{i=1}^{\infty} f_i^{(m)}$$

now

$$\begin{aligned}
\sum_{i=1}^{\infty} f_i^{(m)} &= \sum_{i=1}^{\infty}\sum_{n=1}^{m} P(X = i)1_{n\leq i} \\
&= \sum_{n=1}^{m}\sum_{i=1}^{\infty} P(X = i)1_{n\leq i} && \text{by linearity of the integral} \\
&= \sum_{n=1}^{m} P(X \geq n)
\end{aligned}$$

therefore we have

$$\mathrm{E}\,X = \lim_{m\to\infty} \sum_{n=1}^{m} P(X \geq n)$$

which is the desired result

**(3.2)**

1. For a counting r.v. $X = \sum_{i=1}^{n} 1_{A_i}$, give a formula for the variance of $X$ in terms of the probabilities $P(A_i)$ and $P(A_i \cap P_j), i \neq j$.

   ***answer*** We have by linearity of the expectation,

$$\begin{aligned} \mathrm{E}(X) &= \sum_{i=1}^{n} \mathrm{E}\, 1_{A_i} \\ &= \sum_{i=1}^{n} P(A_i) \end{aligned}$$

   then

$$\begin{aligned} \mathrm{E}(X^2) &= \mathrm{E}(\sum_{i=1}^{n}\sum_{j=1}^{n} 1_{A_i} 1_{A_j}) \\ &= \sum_{i=1}^{n} \mathrm{E}\, 1_{A_i} + 2\sum_{i<j} \mathrm{E}\, 1_{A_i \cap A_j} \\ &= \sum_{i=1}^{n} P(A_i) + 2\sum_{i<j} P(A_i \cap A_j) \end{aligned}$$

   therefore the variance is

$$\begin{aligned} \mathrm{var}(X) &= \mathrm{E}\, X^2 - (\mathrm{E}\, X)^2 \\ &= \sum_{i=1}^{n} P(A_i) + 2\sum_{i<j} P(A_i \cap A_j) - (\sum_{i=1}^{n} P(A_i))^2 \end{aligned}$$

2. If $k$ balls are put at random into $n$ boxes, what is the variance of $X$ = number of empty boxes?

   ***answer*** We can write $X$ as the sum of indicator variables

$$X = \sum_{i=1}^{k} 1_{A_i}$$

   where $A_i$ is the event "box $i$ is empty". Here we have for all $i$, $P(A_i) = (1-1/k)^n$ (all $n$ balls should be put in any box other than the $i$-th one) and similarly for all $i \neq j$, $P(A_i \cap A_j) = (1-2/k)^n$. Applying the above expression, we obtain an expression for the variance

$$\begin{aligned} \mathrm{var}(X) &= \sum_{i=1}^{n} P(A_i) + 2\sum_{i<j} P(A_i \cap A_j) - (\sum_{i=1}^{n} P(A_i))^2 \\ &= n(1-1/k)^n + 2n(n-1)(1-2/k)^n - n^2(1-1/k)^{2n} \end{aligned}$$

**(3.3)**

1. Suppose $\mathrm{E}\, X = 0$ and $\mathrm{var}(X) = \sigma^2 < \infty$. Prove

$$P(X \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

***proof*** Let $c > 0$, and consider the function $\phi : x \mapsto (x+c)^2$. $\phi$ is non-negative, increasing on $(a, \infty)$ (since $a > 0 > -c$).

We have by Chebyshev's inequality

$$\begin{aligned} P(X \geq a) &\leq \frac{\mathrm{E}\,\phi(X)}{\phi(a)} \\ &= \frac{\mathrm{E}\,X^2 + 2c\,\mathrm{E}\,X + c^2}{(a+c)^2} \\ &= \frac{\sigma^2 + c^2}{(a+c)^2} \end{aligned}$$

minimizing the bound over $c$, we have

$$P(X \geq a) \leq \frac{\sigma^2 + c^2}{(a+c)^2}$$

2. Suppose $X \geq 0$ and $\mathrm{E}\,X^2 < \infty$. Prove

$$P(X > 0) \geq \frac{(\mathrm{E}\,X)^2}{\mathrm{E}\,X^2}$$

***proof*** Applying Cauchy-Schwarz inequality to the variables $X$ and $1_{(X>0)}$, we have

$$\mathrm{E}(X 1_{(X>0)}) \leq \sqrt{\mathrm{E}(X^2)\,\mathrm{E}\,1^2_{(X>0)}}$$

or, squaring both sides,

$$(\mathrm{E}\,X)^2 \leq \mathrm{E}(X^2) P(X > 0)$$

which is the desired inequality.

**(3.4 Chebyshev's other inequality)** Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be bounded and increasing functions. Prove that for any r.v. $X$,

$$E(f(X)g(X)) \geq (Ef(X))(E(g(X)))$$

in other words, $f(X)$ and $g(X)$ are positively correlated.

***proof*** Using the definition of the expectation, we can write

$$\begin{aligned} \mathrm{E}\,f(X)\,\mathrm{E}\,g(X) &= \int_\Omega f(X(\omega_1))d\mu(\omega_1) \int_\Omega g(X(\omega_2))d\mu(\omega_2) \\ &= \iint f(X(\omega_1))g(X(\omega_2))d\mu(\omega_1)d\mu(\omega_2) \qquad \text{By Fubini's theorem} \end{aligned}$$

now partitioning $\Omega \times \Omega = A \sqcup A^c$, where $A = \{(\omega_1, \omega_2) \in \Omega \times \Omega : X(\omega_1) \leq X(\omega_2)\}$, and using the fact that on $A$, $f(X(\omega_1)) \leq f(X(\omega_2))$ and on $A^c$, $g(X(\omega_2)) \leq g(X(\omega_1))$ ($f$ and $g$ are increasing)

$$\begin{aligned} \mathrm{E}\,f(X)\,\mathrm{E}\,g(X) &= \iint f(X(\omega_1))g(X(\omega_2))1_A(\omega_1,\omega_2)d\mu(\omega_1)d\mu(\omega_2) + \int f(X(\omega_1))g(X(\omega_2))1_{A^c}(\omega_1,\omega_2)d\mu(\omega_1)d\mu(\omega_2) \\ &\leq \iint f(X(\omega_2))g(X(\omega_2))1_A(\omega_1,\omega_2)d\mu(\omega_1)d\mu(\omega_2) + \int f(X(\omega_1))g(X(\omega_1))1_{A^c}(\omega_1,\omega_2)d\mu(\omega_1)d\mu(\omega_2) \\ &= \int f(X(\omega_2))g(X(\omega_2))h_2(\omega_2)d\mu(\omega_2) + \int f(X(\omega_1))g(X(\omega_1))h_1(\omega_1)d\mu(\omega_1) \end{aligned}$$

where $h_1(\omega_1) = \int_\Omega 1_A(\omega_1, \omega_2) d\mu(\omega_2) = P(X(\omega_1) \leq X)$, and $h_2(\omega_2) = P(X \leq X(\omega_2))$, in particular, for all $\omega$, $h_1(\omega) + h_2(\omega) = 1$. Therefore we have

$$\begin{aligned}
\mathrm{E}\, f(X)\,\mathrm{E}\, g(X) &\leq \int f(X(\omega_2))g(X(\omega_2))h_2(\omega_2)d\mu(\omega_2) + \int f(X(\omega_1))g(X(\omega_1))h_1(\omega_1)d\mu(\omega_1) \\
&= \int f(X(\omega))g(X(\omega))(h_2(\omega) + h_1(\omega))d\mu(\omega) && \text{by renaming the integration var} \\
&= \mathrm{E}\, f(X)g(X)
\end{aligned}$$

which concludes the proof.

**(3.5)** Let $X \sim Poisson(\lambda)$ and $Y \sim Poisson(2\lambda)$.

1. Prove $P(X \geq Y) \leq \exp(-(3 - \sqrt{8})\lambda)$ if $X$ and $Y$ are independent.

   ***proof*** Using a Chernoff type bound, we have for all $\alpha > 0$

   $$\begin{aligned}
   P(X \geq Y) &= P(\alpha^{X-Y} \geq 1) \\
   &\leq \mathrm{E}(\alpha^{X-Y}) && \text{by Markov's inequality} \\
   &= \mathrm{E}\, \alpha^X\, \mathrm{E}\, \frac{1}{\alpha}^Y && \text{by independence}
   \end{aligned}$$

   where

   $$\mathrm{E}\, \alpha^X = \exp(-\lambda) \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \alpha^k = \exp(-\lambda) \exp(\lambda\alpha)$$

   and similarly

   $$\mathrm{E}\, \frac{1}{\alpha}^Y = \exp(-2\lambda) \exp(\frac{2\lambda}{\alpha})$$

   therefore

   $$P(X \geq Y) \leq \exp(\lambda(-3 + \alpha + \frac{2}{\alpha}))$$

   for all $\alpha > 0$. Let $b(\alpha) = \exp(\lambda(-3 + \alpha + \frac{2}{\alpha}))$ be the upper bound. Optimizing the bound over $\alpha$, we have

   $$\begin{aligned}
   b'(\alpha) = 0 &\Leftrightarrow 1 - \frac{2}{\alpha^2} = 0 \\
   &\Leftrightarrow \alpha = \sqrt{2}
   \end{aligned}$$

   thus for $\alpha = \sqrt{2}$, we obtain the desired bound

   $$P(X \geq Y) \leq \exp(\lambda(-3 + 2\sqrt{2}))$$

2. Find constants $A < \infty$, $c > 0$, not depending on $\lambda$, such that without assuming independence, $(X \geq Y) \leq A \exp(-c\lambda)$

***answer*** Since we do not assume independence, we can use Cauchy-Schwarz inequality to bound the expectation of the product. For all $\alpha > 0$, we have

$$\begin{aligned}
P(X \geq Y) &= P(\alpha^{X-Y} \geq 1) \\
&\leq \mathrm{E}(\alpha^{X-Y}) && \text{by Markov's inequality} \\
&\leq \sqrt{\mathrm{E}(\alpha^X)^2\, \mathrm{E}\left(\frac{1}{\alpha}^Y\right)^2} && \text{by Cauchy-Schwarz} \\
&\leq \sqrt{\mathrm{E}(\alpha^2)^X\, \mathrm{E}\,\frac{1}{\alpha^2}^Y}
\end{aligned}$$

The product of expectations is bounded by $b(\alpha^2)$, which is minimal for $\alpha^2 = \sqrt{2}$, and we obtain the following bound

$$P(X \geq Y) \leq \sqrt{\exp(\lambda(-3 + 2\sqrt{2}))} = \exp(\lambda(-\frac{3}{2} + \sqrt{2}))$$

we obtain the desired bound, where $A = 1$ and $c = \frac{3}{2} - \sqrt{2} > 0$.