STAT 205A - Problem Set 02

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(2.1) Let $\mathcal{B}$ be the Borel subsets of $\mathbb{R}$. For $B \in \mathcal{B}$ define

$$\mu(B) = \begin{cases} 1 & \text{if } (0, \epsilon) \subset B \text{ for some } \epsilon > 0 \\ 0 & \text{if not} \end{cases}$$

- Show that μ is not finitely additive on $\mathcal{B}$
- Show hat μ is finitely additive but not countably additive on the field $\mathcal{B}_0$ of finite disjoint unions of intervals $(a, b]$.

answer

- for all $n \geq 1$, let $I_n = \left(\frac{1}{n+1}, \frac{1}{n}\right]$. Now consider the sets $B_0 = \cup_{k=1}^{\infty} I_{2k}$ and $B_1 = \cup_{k=0}^{\infty} I_{2k+1}$. B_0 and B_1 are countable unions of intervals, and are thus Borel sets. Furthermore, B_0 and B_1 are disjoint, $\mu(B_0) = \mu(B_1) = 0$, but $B_0 \sqcup B_1 = (0, 1]$, therefore $\mu(B_0 \cup B_1) = 1$. This shows that μ is not finitely additive on $\mathcal{B}$.
- consider a set $B \in \mathcal{B}_0$ of the form $B = \cup_{i=1}^{n}(a_i, b_i]$ where the union is disjoint. Then $\mu(B) = 1$ if and only if one of the intervals $(a_i, b_i]$ contains $(0, \epsilon)$ for some positive ϵ, i.e. if and only if there exists i such that $0 \in [a_i, b_i)$. Now if we consider two disjoint such sets $B = \cup_{i=1}^{n}(a_i, b_i]$ and $B' = \cup_{j=1}^{n}(a'_i, b'_i]$, then
  - either $\mu(B) = \mu(B') = 0$, in which case $\mu(B \sqcup B' = 0)$
  - or $\mu(B) = 1$ in which case $\mu(B') = 0$ since they are disjoint, in which case $\mu(B \sqcup B') = 1$
  - or $\mu(B') = 1$ in which case $\mu(B) = 0$ since they are disjoint, in which case $\mu(B \sqcup B') = 1$

  This proves that μ is finitely additive on $\mathcal{B}_0$. However, it is not σ-additive. Indeed, consider the sets $I_n = \left(\frac{1}{n+1}, \frac{1}{n}\right]$. Each I_n belongs to $\mathcal{B}_0$, and has measure 0. However, the disjoint countable union $\sqcup_{n=1}^{\infty} I_n = (0, 1]$ has measure 1.

(2.2) Show that, in the definition of "a probability measure μ on a measurable space $(S, \mathcal{S})$", we may replace "countably additive" by "finitely additive, and satisfies

$$\text{if } A_n \downarrow \emptyset \text{ then } \mu(A_n) \downarrow 0$$

proof First note that the two following definitions of σ-additivity are equivalent for probability measures

- $A_n \uparrow A \Rightarrow \mu(A_n) \uparrow \mu(A)$
- $A_n \downarrow A \Rightarrow \mu(A_n) \downarrow \mu(A)$

(one follows from the other by taking complements). Now we show the claim

- Suppose μ is countably additive. Then it is also finitely additive, and if $A_n \downarrow \emptyset$, then by σ-additivity, we have $\mu(A_n) \downarrow \mu(\emptyset) = 0$
- Now suppose that μ is finitely additive and for any nested sequence $A_n \downarrow \emptyset$, $\mu(A_n) \downarrow 0$. Now consider any nested sequence $B_n \downarrow B$, where B is the intersection $B = \cap_n B_n$. For all n, define $A_n = B_n \backslash B$. Then we have A_n is a nested sequence, and $A_n \downarrow \emptyset$ (indeed, $\cap_n A_n = \cap_n (B_n \cap B^c) = (\cap_n B_n) \cap B^c = B \cap B^c = \emptyset$). Therefore $\mu(A_n) \downarrow 0$. But since $B \subset B_n$, we have $B_n = A_n \sqcup B$ where the union is disjoint. Thus by finite additivity, $\mu(B_n) = \mu(A_n) + \mu(B)$. Finally, since $\mu(A_n) \downarrow 0$, we have $\mu(B_n) \downarrow \mu(B)$.

(2.3) Give an example of a measurable space $(S, \mathcal{S})$, a collection $\mathcal{A}$ and probability measures μ and ν such that

- $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$
- $\mathcal{S} = \sigma(\mathcal{A})$
- $\mu \neq \nu$

answer Consider $S = \{1,2,3,4\}$, $\mathcal{S} = P(S)$, $\mathcal{A} = \{\{1,2\},\{2,3\}\}$. Note that we have $\sigma(\mathcal{A}) = \mathcal{S}$. Now define μ and ν by

- $\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = 1/3$ and $\mu(\{4\}) = 0$ (this completely determines μ by finite additivity)
- $\nu(\{1\}) = \nu(\{3\}) = 0$, $\nu(\{2\}) = 2/3$ and $\nu(\{4\}) = 1/3$ (this completely determines ν by finite additivity).

Then we have $\mu(\{1,2\}) = \nu(\{1,2\}) = 2/3$ and $\mu(\{2,3\}) = \nu(\{2,3\}) = 2/3$, i.e. μ and ν agree on $\mathcal{A}$, but μ and ν are different.

(2.4) Let μ be a probability measure on $(S, \mathcal{S})$, where $\mathcal{S} = \sigma(\mathcal{F})$ for a field $\mathcal{F}$. Show that for each $B \in \mathcal{S}$ and $\epsilon > 0$, there exists $A \in \mathcal{F}$ such that $\mu(B \Delta A) < \epsilon$

proof Since $\mathcal{F}$ is a field, it is in particular closed under intersection, thus it forms a π-class of subsets of S. Now let $\mathcal{B}$ be the collection of subsets

$$\mathcal{B} = \{B \subset S : \forall \epsilon > 0, \exists A \in \mathcal{F} : \mu(B \Delta A) < \epsilon$$

it suffices to show that $\mathcal{B}$ is a λ-class, then by Dynkin's Lemma, we have $\sigma(\mathcal{F}) \subseteq \mathcal{B}$, i.e. $\mathcal{S} \subseteq \mathcal{B}$, which proves the claim.

Now let us show that $\mathcal{B}$ is a λ-class that contains $\mathcal{F}$:

- we have $S \in \mathcal{B}$: indeed, for all $\epsilon > 0$, we have $\mu(S \Delta S) = \mu(\emptyset) = 0 < \epsilon$, and $S \in \mathcal{F}$ (since $\mathcal{F}$ is a field)
- let $B_1, B_2 \in \mathcal{B}$. We seek to show that $B = B_1 \setminus B_2 \in \mathcal{B}$. Fix $\epsilon > 0$. There exist A_1, A_2 such that $\mu(B_1 \Delta A_1) < \epsilon/2$ and $\mu(B_2 \Delta A_2) < \epsilon/2$. Now consider $A = A_1 \setminus A_2$. We have

  $$B \Delta A = (B \cap A^c) \cup (B^c \cap A)$$

  the first term is $B \cap A^c = (B_1 \cap B_2^c) \cap (A_1^c \cup A_2) = (B_1 \cap B_2^c \cap A_1^c) \cup (B_1 \cap B_2^c \cap A_2) \subseteq (B_1 \cap A_1^c) \cup (B_2^c \cap A_2) \subseteq (B_1 \Delta A_1) \cup (B_2 \Delta A_2)$. Similarly, the second term is also a subset of $(B_1 \Delta A_1) \cup (B_2 \Delta A_2)$ (by symmetry in A, B). Therefore

  $$B \Delta A \subseteq (B_1 \Delta A_1) \cup (B_2 \Delta A_2)$$

  and its measure is $\leq \mu((B_1 \Delta A_1) \cup (B_2 \Delta A_2)) \leq \mu(B_1 \Delta A_1) + \mu(B_2 \Delta A_2) < \epsilon$.

- Let B_n be a nested sequence of elements of $\mathcal{B}$, such that $B_n \uparrow B$. We seek to show that $B \in \mathcal{B}$. Fix $\epsilon > 0$. Since $B_n \uparrow B$, then $\mu(B_n) \uparrow \mu(B)$. Thus there exists N such that $\mu(B_N) > \mu(B) - \epsilon/2$. Since $B_N \in \mathcal{B}$, there exists $A \in \mathcal{F}$ such that $\mu(B_N \Delta A) \leq \epsilon/2$. Now consider $B\Delta A$. We have can write $B = B_N \cup Z$ where $Z = B \setminus B_N$ and $\mu(Z) < \epsilon/2$. Then we have

$$\begin{aligned} B\Delta A &= (B^c \cap A) \cup (B \cap A^c) \\ &= (B_N^c \cap Z^c \cap A) \cup ((B_N \cup Z) \cap A^c) \\ &\subseteq (B_N^c \cap A) \cup (B_N \cap A^c) \cup (Z \cap A^c) \\ &\subseteq (B_N \Delta A) \cup Z \end{aligned}$$

therefore

$$\mu(B\Delta A) \leq \mu(B_N \Delta A) + \mu(Z) < \epsilon/2 + \epsilon/2$$

and it follows that $B \in \mathcal{B}$

Finally, $\mathcal{B}$ contains $\mathcal{F}$ since for all $B \in \mathcal{F}$ and for all $\epsilon > 0$, $\mu(B\Delta B) = \mu(\emptyset) = 0 < \epsilon$. This concludes the proof.

(2.5) Let $g : [0,1] \to \mathbb{R}$ be integrable w.r.t. Lebesgue measure. Let $\epsilon > 0$. Show that there exists a continuous function $f : [0,1] \to \mathbb{R}$ such that $\int |f(x) - g(x)| dx \leq \epsilon$.

proof We first show that there exists a simple function $s : [0,1] \to \mathbb{R}$ such that $\int |s(x) - g(x)| \leq \epsilon/2$. This follows from the definition of the integral of an integrable function

$$\int g d\mu = \int g^+ d\mu - \int g^- d\mu$$

where $\int g^+ d\mu = \sup \int_{s \text{ simple}, s \leq g^+} s d\mu$ (and similarly for g^-, both functions being non-negative). Therefore there exist simple functions s_1 and s_2 such that

$$\begin{aligned} &s_1 \leq g^+ \\ &s_2 \leq g^- \\ &\int s_1 \geq \int g^+ d\mu - \epsilon/4 \\ &\int s_2 \geq \int g^- d\mu - \epsilon/4 \end{aligned}$$

define a simple function $s = s_1 - s_2$. Then we have

$$\begin{aligned} \int |s - g| d\mu &= \int |s_1 - s_2 - g^+ + g^-| \\ &\leq \int |s_1 - g^+| d\mu + \int |s_2 - g^-| d\mu && \text{by the triangle inequality} \\ &\leq \epsilon/4 + \epsilon/4 && \text{by definition of } s_1,\ s_2 \end{aligned}$$

Now it suffices to show that any simple function s defined on $[0,1]$ can be approximated by a continuous function on $[0,1]$, in the following sense: there exists $f : [0,1] \to \mathbb{R}$ continuous, such that $\int |s - f| \leq \epsilon/2$. We can write s as the finite sum of scaled indicator functions

$$s = \sum_{i=1}^{n} c_i 1_{A_i}$$

where for all i, $A_i \subseteq [0,1]$ is a Lebesgue measurable set, and the A_i's are disjoint. By regularity of the Lebesgue measure, there exist a compact set K_i and an open set O_i such that

$$\begin{aligned} &K_i \subseteq A_i \subseteq O_i \\ &\mu(A_i \setminus K_i) \leq \frac{\epsilon}{4|c_i|n} \\ &\mu(O_i \setminus A_i) \leq \frac{\epsilon}{4|c_i|n} \end{aligned}$$

in particular, we have $\mu(O_i \cap K_i^c) \leq \frac{\epsilon}{2|c_i|n}$. Now consider the closed sets O_i^c and K_i. By Urysohn's Lemma, there exists a continuous function $f_i : [0,1] \to \mathbb{R}$ such that f_i is identically 0 on O_i^c, and identically 1 on K_i, therefore

- $|f_i(x) - 1_{A_i}(x)| = 0$ for all $x \in O_i^c \cup K_i$
- $|f_i(x) - 1_{A_i}(x)| \leq 1$ for all $x \in (O_i^c \cup K_i)^c = O_i \cap K_i^c$
- $\int |f_i - 1_{A_i}| d\mu \leq \mu(O_i \cap K_i^c) \leq \frac{\epsilon}{2|c_i|n}$

Now consider the function $f = \sum_{i=1}^n c_i f_i$. We have f is continuous as the finite sum of continuous functions, and

$$\begin{aligned} \int |s - f| d\mu &= \int |\sum_{i=1}^n c_i(1_{A_i} - f_i)| d\mu \\ &\leq \sum_{i=1}^n |c_i| \int |f_i - 1_{A_i}| \\ &\leq \sum_{i=1}^n |c_i| \frac{\epsilon}{2|c_i|n} \\ &= \epsilon/2 \end{aligned}$$

This concludes the proof.