STAT 205A - Problem Set 01

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(1.1) Let $\mathcal{F}_n$ be collections of subsets of a set S. Suppose that each $\mathcal{F}_n$ is a field, and $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all n. Define $\mathcal{F} = \cup_n \mathcal{F}_n$. Show that $\mathcal{F}$ is a field. Give an example to show that $\mathcal{F}$ need not be a σ-field.

answer $\mathcal{F}$ is nonempty since $\mathcal{F}_1$ is non-empty. It suffices to check that $\mathcal{F}$ is closed under union and complement.

- Let $A \in \mathcal{F}$. Then $\exists n$ such that $A \in \mathcal{F}_n$, and since $\mathcal{F}_n$ is a field, $A^c \in \mathcal{F}_n \subset \mathcal{F}$.
- Let $A_1, A_2 \in \mathcal{F}$. Then there exist n_1, n_2 such that $A_1 \in \mathcal{F}_{n_1}$, $A_2 \in \mathcal{F}_{n_2}$; assume without loss of generality that $n_1 \leq n_2$. Then since $\mathcal{F}_{n_1} \subset \mathcal{F}_{n_2}$, A_1 and A_2 are both elements of F_{n_2}, and since $\mathcal{F}_{n_2}$ is a field, $A \cup B \in \mathcal{F}_{n_2} \subset \mathcal{F}$.

Therefore $\mathcal{F}$ is a field. $\mathcal{F}$ is not necessarily a σ-field. Indeed, consider the case where $S = \mathbb{R}$ and $\mathcal{F}_n = \mathcal{F}(\mathcal{A})$ for all n, the field generated by $\mathcal{A}$, the set of open intervals of the form $(-\infty, a)$, $a \in \mathbb{R}$. Then the union $\mathcal{F}$ is a field, but it is not a σ-field, since the σ-field generated by $\mathcal{A}$ contains for example countable unions of disjoint open intervals, and these sets are not in $\mathcal{F}$.

(1.2) Given a nonempty collection $\mathcal{A}$ of sets, we defined $\mathcal{F}(\mathcal{A})$ as the intersection of all fields containing $\mathcal{A}$. Show that $\mathcal{F}(\mathcal{A})$ is the class of sets of the form

$$\cup_{i=1}^{m} \cap_{j=1}^{n_i} A_{i,j} \tag{1}$$

where for each i and j, either $A_{i,j} \in \mathcal{A}$ or $A_{i,j}^c \in \mathcal{A}$, and the sets $\cap_{j=1}^{n_i} A_{i,j}$, $i \in \{1, \ldots, m\}$ are pairwise disjoints.

proof Let $\mathcal{B}$ be this class of sets. $\mathcal{B}$ is a subset of any field that contains $\mathcal{A}$, therefore $\mathcal{B} \subseteq \mathcal{F}(\mathcal{A})$. It remains to show that $\mathcal{B}$ is a field.

- First, $\mathcal{B}$ contains the empty set: take any element $A \in \mathcal{A}$, then set $A_{1,1} = A$, $A_{1,2} = A^c$. Then by definition, $\mathcal{B}$ contains $A_{1,1} \cap A_{1,2} = \emptyset$ (taking $n = 1$ in the expression (1)).
- $\mathcal{B}$ is closed under intersection: let $B = \cup_i \cap_j A_{i,j}$ and $B' = \cup_k \cap_l A'_{k,l}$ be two elements of $\mathcal{B}$, and call $B_i = \cap_j A_{i,j}$ and $B'_k = \cap_l A'_{k,l}$. By assumption, the B_i's are pairwise disjoint, and so are the B'_k's. Then we have
  $$B \cap B' = \cup_{i,k} (B_i \cap B'_k)$$
  where the sets $B_i \cap B'_k$ are pairwise disjoint, and can be further expanded into an intersection of the form
  $$B_i \cap B'_k = (\cap_j A_{i,j}) \cap (\cap_l A'_{k,l})$$
  where by assumption, each $A_{i,j}$ and $A'_{k,l}$ is either an element of $\mathcal{A}$ or has its complement in $\mathcal{A}$.
- $\mathcal{B}$ is closed under complement: let $B = \cup_i \cap_j A_{i,j}$. Then
  $$B^c = \cap_i \cup_j A_{i,j}^c$$
  for each i, $\cup_j A_{i,j}^c$ is in $\mathcal{B}$ since it is of the form 1 (taking n_i to be 1). Finally, since $\mathcal{B}$ is closed under intersection, the intersection $\cap_i \cup_j A_{i,j}^c$ is in $\mathcal{B}$. This concludes the proof.

(1.3) Suppose $B \in \sigma(\mathcal{A})$ for some collection $\mathcal{A}$ of subsets. Show there exists a countable subcollection $\mathcal{A}_B \subset \mathcal{A}$ such that $B \in \sigma(\mathcal{A}_B)$.

proof Consider the union

$$\mathcal{B} = \cup_{\mathcal{C} \subset \mathcal{A} : \mathcal{C} \text{is countable}} \sigma(\mathcal{C}) \tag{2}$$

the claim is that $\sigma(\mathcal{A}) \subset \mathcal{B}$. To prove this, it suffices to show that $\mathcal{B}$ is a σ-field that contains all elements of $\mathcal{A}$.

- Let $A \in \mathcal{A}$. We have $\{A\}$ is a countable (finite) subset of $\mathcal{A}$, thus $\sigma(\{A\}) \subset \mathcal{B}$, therefore $A \in \sigma(\{A\}) \subset \mathcal{B}$ which proves that all elements of $\mathcal{A}$ are in $\mathcal{B}$
- $\mathcal{B}$ is closed under countable union: consider a sequence (B_n) of elements of $\mathcal{B}$. By definition, for each n, there exists $\mathcal{C}_n$ countable subset of $\mathcal{A}$ such that $B_n \in \sigma(\mathcal{C})$. Now let $\mathcal{C} = \cup_n \mathcal{C}_n$, which is also a countable subset of $\mathcal{A}$ (a countable union of countable sets is countable) thus $\sigma(\mathcal{C}) \subset \mathcal{B}$. And since $\mathcal{C}$ is a σ-field and contains each B_n, it also contains the countable union $\cup_n B_n$. This proves that $\cup_n B_n \in \mathcal{B}$.
- $\mathcal{B}$ is closed under complement: let $B \in \mathcal{B}$. Then there exists $\mathcal{C}$ countable subset of $\mathcal{A}$ such that $B \in \sigma(\mathcal{C})$. By definition of a σ-field, $\sigma(\mathcal{C})$ also contains the complement B^c, therefore $B^c \in \sigma(\mathcal{C}) \subset \mathcal{B}$.

This completes the proof that $\mathcal{B}$ is a σ-field that contains all elements of $\mathcal{A}$.

(1.4) Show that the Borel σ-field on $\mathbb{R}^d$ is the smallest σ-field that makes all continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ measurable.

proof Let $\mathcal{B}$ be the Borel σ-field on $\mathbb{R}^d$ (smallest σ-field that contains open sets), and let $\mathcal{C}$ be the smallest σ-field that makes all continuous functions measurable. The claim is that $\mathcal{B} = \mathcal{C}$.

First, let f be a continuous function. Then f is measurable for $\mathcal{B}$ (it suffices to show that $f^{-1}((-\infty, a))$ is measurable for any $a \in \mathbb{R}$, but since f is continuous the inverse image of open sets is open, thus in $\mathcal{B}$). Therefore $\mathcal{B}$ is a σ-field that makes all continuous functions measurable, therefore $\mathcal{C} \subseteq \mathcal{B}$.

To show the reverse inclusion, it suffices to show that $\mathcal{C}$ contains closed sets. Let C be a closed set in $\mathbb{R}^d$, and consider the function

$$\begin{aligned} f_C : \mathbb{R}^d &\to \mathbb{R} \\ x &\mapsto d(x, C) \end{aligned}$$

where the distance is defined as $d(x, C) = \inf_{y \in C} \|x - y\|_2$ for example. Here the inf is in fact attained, but this property is not needed. f_C is continuous, since $|f(x_1) - f(x_2)| \leq \|x_1 - x_2\|_2$ for all $x_1, x_2 \in C$. [1] Therefore $\mathcal{C}$ contains $f^{-1}(\{0\})$, which is exactly C. Therefore $\mathcal{C}$ contains all closed sets, and since it is a σ-field, we must have $\mathcal{B} \subset \mathcal{C}$.

(1.5) A function $f : \mathbb{R}^d \to \mathbb{R}$ is lower semicontinuous (l.s.c.) if $\liminf_{y \to x} f(y) \leq f(x)$ for all x. A function is upper semicontinuous (u.s.c.) if $\limsup_{y \to x} f(y) \leq f(x)$ for all x. Show that if f is l.s.c. or u.s.c., then f is measurable.

[1] one way to show this is by definition of the distance: for all $\epsilon > 0$, there exists y_1 such that $f_C(x_1) \geq \|x_1 - y_1\| - \epsilon$. Then

$$\begin{aligned} f_C(x_2) &\leq \|x_2 - y_1\| \\ &\leq \|x_2 - x_1\| + \|x_1 - y_1\| && \text{by the triangle inequality} \\ &\leq \|x_2 - x_1\| + f_C(x_1) + \epsilon \end{aligned}$$

since this holds for any $\epsilon > 0$, we have $f_C(x_2) \leq \|x_2 - x_1\| + f_C(x_1)$. Then by symmetry in x_1, x_2, it follows that $|f_C(x_2) - f_C(x_1)| \leq \|x_2 - x_1\|_2$.

proof Let f be lower semicontinuous. To show that f is measurable, it suffices to show that $f^{-1}((-\infty, a))$ is a measurable set for all $a \in \mathbb{R}$. Let $a \in \mathbb{R}$, and call $F_a = f^{-1}((-\infty, a))$. The claim is that F_a is an open set. Indeed, let $x \in F_a$. Then $f(x) < a$. Now since f is l.s.c., we have $\limsup_{y \to x} f(y) \leq f(x)$, thus for all $\epsilon > 0$, there exists $\delta > 0$ such that $\inf_{y \in B(x,\delta)} f(y) < \limsup_{y \to x} f(y) + \epsilon \leq f(x) + \epsilon$. In particular for $\epsilon = a - f(x)$, there exists $\delta > 0$ such that $\sup_{y \in B(x,\delta)} f(y) < f(x) + \epsilon = a$, i.e. $B(x, delta) \subset F_a$. This proves that every point in F_a is in the interior of F_a, i.e. that F_a is open, hence F_a is measurable, which completes the proof.

For the case of upper semi-continuous functions, consider the function $-f$, which is lower semicontinuous. By the previous argument, $-f$ is measurable, and so is f (product of two measurable functions is measurable).