Prediction, Learning and Games - Chapter 7
Prediction and Playing Games

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Review

- K players
- player k: N_k actions

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- joint action vector $i \in [N_1] \times \cdots \times [N_K]$
- loss vector $\ell(i) \in [0,1]^K$

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- K players
- player k: N_k actions
- joint action vector $i \in [N_1] \times \cdots \times [N_K]$
- loss vector $\ell(i) \in [0, 1]^K$
- players randomize: mixed strategy profile $\pi \in \Delta^{[N_1]} \times \cdots \times \Delta^{[N_K]}$

$$\pi(i) = p^{(1)}(i_1) \dots p^{(K)}(i_K)$$

- joint random action $I \in [N_1] \times [N_K]$, $I \sim \pi$
- expected loss $E_{I \sim \pi}[\ell^{(k)}] = \sum_i \pi(i) \ell^{(k)}(i)$, denoted $\ell^{(k)}(\pi)$

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- Define
  product of average marginals

$$\bar{p}_t^{(1)} \times \cdots \times \bar{p}_t^{(K)}$$

  and average joint

$$\bar{P}_t = \overline{p^{(1)} \times \cdots \times p^{(K)}}$$

Review

Nash equilibria $\mathcal{N}$

for all k and all p'

$$\ell^{(k)}(p^{(k)}, p^{(-k)}) \leq \ell^{(k)}(p', p^{(-k)})$$

Correlated equilibria $\mathcal{C}$

distribution P over $[N_1] \times \cdots \times [N_K]$, which is not necessarily a product distribution

For all j, j'

$$\sum_{i:i^k=j} P(i)\ell^{(k)}(j, i^{(-k)}) \leq \sum_{i:i^k=j} P(i)\ell^{(k)}(j', i^{(-k)})]$$

Hannan set $\mathcal{H}$

For all j

$$\sum_{i} P(i)\ell^{(k)}(i^{(k)}, i^{(-k)}) \leq \sum_{i} P(i)\ell^{(k)}(j, i^{(-k)})\}$$

Summary of results

- $\mathcal{N}$ Nash equilibria
- $\mathcal{C}$ Correlated equilibria
- $\mathcal{H}$ Hannan set

$$\mathcal{N} \subset \mathcal{C} \subset \mathcal{H}$$

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- $\mathcal{N}$ Nash equilibria
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- Minimizing regret:
  - $(\bar{p}_t^{(1)} \times \cdots \times \bar{p}_t^{(K)}) \to \mathcal{N}$
  - $(p_t^{(1)} \times \cdots \times p_t^{(K)}) \to \mathcal{N}$?
  - $\bar{P}_t \to \mathcal{H}$
- Minimizing internal regret:
  - $\bar{P}_t \to \mathcal{C}$, updates $O(N^2)$

Blackwell approachability

- What is approachability?

Blackwell approachability

- What is approachability?
- 2 player game, where losses are vector valued
- Assume everything remains in the unit ball, $\|\ell(i,j)\|_2 \leq 1$
- A closed convex set $S \subset B(0,1)$ is approachable if $\exists$ a randomized strategy for (row) such that for all sequences of (column),

$$\lim_{T\to\infty} d\left(\frac{1}{T}\sum_{t\leq T}\ell(I_t, J_t), S\right) = 0$$

Blackwell approachability

Zero-sum 2 player games

Scalar valued. Sets that are approachable:

$$(-\infty, c]$$

with $c \geq V$

Blackwell approachability

Zero-sum 2 player games

Scalar valued. Sets that are approachable:

$$(-\infty, c]$$

with $c \geq V$

- $(-\infty, V]$ is approachable: (row) minimizes regret
- $(-\infty, V - \epsilon]$ is not approachable: (column) minimizes regret

Blackwell approachability: Half spaces

- $H_{a,c} = \{\ell : \langle a, \ell \rangle \leq c\}$

Blackwell approachability: Half spaces

- $H_{a,c} = \{\ell : \langle a, \ell \rangle \leq c\}$
- Define new game: loss of (i, j) is $\langle a, \ell(i,j) \rangle := (M_a)_{i,j}$. Defines a matrix M_a
- This a scalar game

Blackwell approachability: Half spaces

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Half spaces

$H_{a,c}$ is approachable iff $c \geq V = \min_{p \in \Delta} \max_{q \in \Delta} \langle p, M_a q \rangle$

- in particular, if $H_{a,c}$ is approachable, then $\exists p \in \Delta : \max_q \langle p, M_a q \rangle \leq c$
- Call it $p^*(H_{a,c})$.
- Note that by playing $p^*(H_{a,c})$, you are guaranteed to be in $H_{a,c}$ at every step

Blackwell approachability theorem

Theorem

Blackwell Approachability

S is approachable iff every half space containing S is approachable.

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Blackwell Approachability
S is approachable iff every half space containing S is approachable.

- $\Rightarrow$ if S is approachable, any super set is approachable.
- $\Leftarrow$ Assume every half space containing S is approachable.
  - average loss $A_t = \frac{1}{t}\sum_{\tau \leq t} \ell(i_\tau, j_\tau)$
  - at t, if $A_{t-1} \notin S$, project $\pi_S(A_{t-1})$, define half-space H_{t-1}: through $\pi_S(A_{t-1})$, $\perp (A_{t-1} - \pi_S(A_{t-1}))$, containing S.
  - By assumption H_{t-1} is approachable. Play $p^*(H_{t-1})$
  - on the next step: $\ell(p^*(H_{t-1}), j_t) \in H_{t-1}$

Blackwell approachability theorem

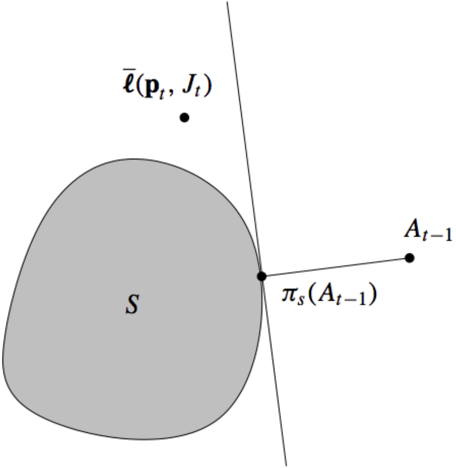

Figure : expected loss $\bar{\ell}(p_t, J_t)$ is forced in H_{t-1}

Blackwell approachability theorem

$$\begin{aligned}
d(A_t, S)^2 &\leq \|A_t - \pi_S(A_{t-1})\|_2^2 \\
&= \|\frac{t-1}{t}A_{t-1} + \frac{1}{t}\ell(p_t, j_t) - (\frac{t-1}{t} + \frac{1}{t})\pi_S(A_{t-1})\|_2^2 \\
&= \left(\frac{t-1}{t}\right)^2 d(A_{t-1}, S)^2 + \frac{1}{t^2}\|\ell(p_t, j_t) - \pi_S(A_{t-1})\|_2^2 \\
&\quad + 2\frac{t-1}{t^2}\langle A_{t-1} - \pi_S(A_{t-1}), \ell(p_t, j_t) - \pi_S(A_{t-1})\rangle \\
&\leq \left(\frac{t-1}{t}\right)^2 d(A_{t-1}, S)^2 + \frac{1}{t^2}2^2
\end{aligned}$$

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&= \left(\frac{t-1}{t}\right)^2 d(A_{t-1}, S)^2 + \frac{1}{t^2}\|\ell(p_t, j_t) - \pi_S(A_{t-1})\|_2^2 \\
&\quad + 2\frac{t-1}{t^2}\langle A_{t-1} - \pi_S(A_{t-1}), \ell(p_t, j_t) - \pi_S(A_{t-1})\rangle \\
&\leq \left(\frac{t-1}{t}\right)^2 d(A_{t-1}, S)^2 + \frac{1}{t^2}2^2
\end{aligned}
$$

multiply by t^2, then by induction

$$t^2 d(A_t, S)^2 \leq 4t$$

thus

$$d(A_t, S) \leq \frac{2}{\sqrt{t}}$$

Blackwell approachability theorem

- Not quite what we want, since we proved it for expected losses.
- In fact, we have almost sure convergence to S.
- Additional argument needed, see the book.

Potential based approachability

- S approachable. Want to define a class of algorithms which approach S.
- potential $\phi \geq 0$ convex, $\phi(x) = 0 \Leftrightarrow x \in S$.
- e.g. $\phi(s) = d(x, S)^2$

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Algorithm

Same idea: if $A_{t-1} \notin S$, use the potential to choose a strategy p_t such that

$$\max_j \left\langle a_{t-1}, \bar{\ell}(p_t, j) \right\rangle \leq c_{t-1}$$

where

$$a_{t-1} = \nabla\phi(A_{t-1}) \qquad c_{t-1} = \sup_{x \in S} \langle a_{t-1}, x \rangle$$

Potential based approachability

Bregman projection

Same as previous problem, with

$$\pi_S(\ell) = \arg\max_{x \in S} \langle \nabla \phi(\ell), x \rangle$$