MATH 202B - Problem Set 9

Walid Krichene (23265217)

April 17, 2013

(9.1) Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner product space. Show that if two nonzero vectors $x, y \in \mathcal{H}$ satisfy $|\langle x, y\rangle| = \|x\|\|y\|$, then there exists $c \in \mathbb{F}$ such that $y = cx$.

proof We have for all $c \in \mathbb{C}$,

$$\begin{aligned}\|y - cx\|_2^2 &= \langle y - cx, y - cx\rangle \\ &= \|y\|^2 + \|cx\|^2 - \langle y, cx\rangle - \langle cx, y\rangle \\ &= \|y\|^2 + |c^2|\|x\|^2 - 2\operatorname{Re}(c\langle x, y\rangle)\end{aligned}$$

consider in particular $c = \frac{\overline{\langle x,y\rangle}}{\|x\|^2}$ (well defined since x is a nonzero vectors). Then

$$\begin{aligned}\|y - cx\|^2 &= \|y^2\| + \frac{|\langle x, y\rangle|^2}{\|x\|^4}\|x\|^2 - 2\operatorname{Re}\left(\frac{\overline{\langle x, y\rangle}}{\|x\|^2}\langle x, y\rangle\right) \\ &= \|y^2\| + \frac{|\langle x, y\rangle|^2}{\|x\|^2} - 2\frac{|\langle x, y\rangle|^2}{\|x\|^2} \\ &= \|y^2\| - \frac{|\langle x, y\rangle|^2}{\|x\|^2} \\ &= 0\end{aligned}$$

which proves that $y = cx$

(9.2) Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\phi_n, n \in \mathbb{N}$ be an orthonormal basis for $L^2(X) = L^2(X, \mathcal{A}, \mu)$. This is a space of equivalence classes of complex-valued measurable functions. Define functions

$$\begin{aligned}\phi_m \otimes \phi_n : X \times X &\to \mathbb{C} \\ (x, y) &\mapsto \phi_m(x)\phi_n(x)\end{aligned}$$

Show that $\mathcal{O} = \{\phi_m \otimes \phi_n, (m, n) \in \mathbb{N}^2\}$ is an orthonormal basis for $L^2(X \times X, \mathcal{A} \times \mathcal{A}, \mu \times \mu)$.

proof We show that $\mathcal{O}$ is an orthonormal set of $L^2(X \times X)$ functions, and that for all $f \in L^2(X \times X)$, $\|f\|^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\langle f, \phi_m \otimes \phi_n\rangle|^2$. We have

- for all m, n, we have

$$\begin{aligned}
\|\phi_n \otimes \phi_m\|^2 &= \int_{X\times X} |\phi_n \otimes \phi_m|^2 d\mu_1 \times \mu_2 \\
&= \int_X \int_X |\phi_n(x)|^2 |\phi_m(y)|^2 d\mu_2(y)\mu_1(x) && \text{by Fubini's theorem} \\
&= \int_X |\phi_n(x)|^2 d\mu_1(x) \int_X |\phi_m(y)|^2 d\mu_2(y) \\
&= \|\phi_n\|^2 \|\phi_m\|^2
\end{aligned}$$

which is finite since $\phi_n, \phi_m \in L^2(X)$. Thus $\phi_n \otimes \phi_m \in L^2(X \times X)$.

- we have for all m, n, i, j

$$\begin{aligned}
\langle \phi_m \otimes \phi_n, \phi_i \otimes \phi_j \rangle &= \int_{X\times X} \phi_n(x)\phi_m(y)\phi_i(x)\phi_j(y) d\mu_1 \times \mu_2(x, y) \\
&= \int_X \left(\int_X \phi_n(x)\phi_i(x)\phi_m(y)\phi_j(y) d\mu_2(y) \right) d\mu_1(x) && \text{by Fubini's theorem} \\
&= \int_X \phi_n(x)\phi_i(x) d\mu_1(x) \int_X \phi_m(y)\phi_j(y) d\mu_2(y) \\
&= \langle \phi_m, \phi_j \rangle \langle \phi_n, \phi_i \rangle \\
&= \begin{cases} 1 & \text{if } (m, n) = (i, j) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

thus $\mathcal{O}$ is an orthonormal set.

- now let $f \in L^2(X \times X)$. We have

$$\begin{aligned}
\langle f, \phi_n \otimes \phi_m \rangle &= \int_{X\times X} f(x, y)\overline{\phi_n(x)\phi_m(y)} d\mu_1 \times \mu_2(x, y) \\
&= \int_X \left(\int_X f^x(y)\overline{\phi_m(y)} d\mu_2(y) \right) \overline{\phi_n(x)} d\mu_1(x) && \text{by Fubini's theorem} \\
&= \int_X \langle f^x, \phi_m \rangle \overline{\phi_n(x)} d\mu_1(x) \\
&= \langle \pi_m, \phi_n \rangle
\end{aligned}$$

where $\pi_m : x \mapsto \langle f^x, \phi_n \rangle$. Next, we have

$$\begin{aligned}
\|f\|^2_{L^2(X\times X)} &= \int_{X\times X} |f|^2 d\mu_1 \times \mu_2 \\
&= \int_X \left(\int_X |f^x(y)|^2 d\mu_2(y) \right) d\mu_1(x) \\
&= \int_X \|f^x\|^2 d\mu_1(x)
\end{aligned}$$

and in particular, $f^x \in L^2(X)$ for a.e. x (otherwise $\|f\|^2 = \infty$). Let A be the set of x such that $f^x \in L^2$. For all $x \in A$, since $\{\phi_n, n \in \mathbb{N}\}$ is an orthonormal basis of $L^2(X)$, we have

$$\|f^x\|^2 = \sum_{m=1}^{\infty} |\langle f^x, \phi_m \rangle|^2 = \sum_{m=1}^{\infty} |\pi_m(x)|^2$$

therefore

$$\begin{aligned}\|f\|^2 &= \int_A \sum_{m=1}^{\infty} |\pi_m(x)|^2 d\mu_1(x) \\ &= \sum_{m=1}^{\infty} \int_A |\pi_m(x)|^2 d\mu_1(x) && \text{by the DCT} \\ &= \sum_{m=1}^{\infty} \|\pi_m\|^2\end{aligned}$$

here each π_m is in $L^2(X)$ (the sum is finite), thus $\|\pi_m\|^2 = \sum_{n=1}^{\infty}\langle \pi_m, \phi_n\rangle$. Combining this with the last equality, we have

$$\begin{aligned}\|f\|^2 &= \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}}\langle \pi_m, \phi_n\rangle \\ &= \sum_{m\in\mathbb{N}, n\in\mathbb{N}} \langle \pi_m, \phi_n\rangle && \text{by Fubini's theorem} \\ &= \sum_{m\in\mathbb{N}, n\in\mathbb{N}} \langle f, \phi_n \otimes \phi_m\rangle\end{aligned}$$

as claimed

(9.3) Let $\mathcal{H}$ be a Hilbert space, and let $L \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ be an isometry in the sense that $\|L(x)\| = \|x\|$ for every $x \in \mathcal{H}$.

1. Prove that the range of L is a closed subspace of $\mathcal{H}$.

   proof The range of L is a subspace since L is linear (if $y_1, y_2 \in L(\mathcal{H})$, then there exist $x_1, x_2 \in \mathcal{H}$ such that $L(x_1) = y_1$ and $L(x_2) = y_2$. Then $y_1 + y_2 = L(x_1 + x_2) \in L(\mathcal{H})$, and for all $\lambda \in \mathcal{F}$, $\lambda y = L(\lambda x) \in L(\mathcal{H})$)

   Now let y be a vector in the closure of $L(\mathcal{H})$. Then there exists a sequence $(y_n)_n$ of elements of $L(\mathcal{H})$ that converges to y. For all n, there exists $x_n \in \mathcal{H}$ such that $y_n = L(x_n)$. Then (x_n) is de Cauchy since

   $$\|x_n - x_m\| = \|L(x_n - x_m)\| = \|L(x_n) - L(x_m)\| = \|y_n - y_m\|$$

   and (y_n) is Cauchy. Since the space is Hilbert, (x_n) converges. Let x be its limit. Then we have

   $$\|L(x) - L(x_n)\| = \|x - x_n\|$$

   which converges to zero as $n \to \infty$, thus $L(x) = \lim_n L(x_n) = \lim_n y_n = y$. Therefore $L(x) = y$ and $y \in L(\mathcal{H})$.

2. Give an example of an isometry L which is not surjective.

   answer Consider the space ℓ^2 of complex sequences (a_n) such that $\sum_{n\in\mathbb{N}} |a_n|^2 < \infty$, and consider the linear operator that shifts the sequence to the right

   $$\begin{aligned} L : \ell^2 &\to \ell^2 \\ (a_1, a_2, \dots) &\mapsto (0, a_1, a_2, \dots)\end{aligned}$$

   L is clearly linear, an isometry, and the sequence $(1, 0, 0, \dots)$ is not in the range of L.

(9.4) Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\mathcal{H} = L^2(X, \mathcal{A}, \mu)$, equivalence classes of complex-valued functions.

1. Show that for any function $a \in L^\infty(X)$,

$$\begin{aligned} L_a &: \mathcal{H} \to \mathcal{H} \\ & f \mapsto L_a(f) = x \mapsto a(x)f(x) \end{aligned}$$

defines an operator in $\mathcal{B}(\mathcal{H}, \mathcal{H})$.

proof L_a is clearly linear, since

$$\begin{aligned} L_a(f+g) &= a(f+g) = af + ag = L_a(f) + L_a(g) \\ L(\lambda f) &= a\lambda f = \lambda a f = \lambda L_a(f) \end{aligned}$$

We verify that for all $a \in L^\infty(X)$ and all $f \in L^2(X)$, af is indeed in $L^2(X)$. Let

$$M = \|a\|_\infty = \inf\{N > 0 : \mu(\{x : |a(x)| > N\}) = 0\}$$

be the essential supremum of a. Then for a.e. x, $|a(x)| \leq M$. Let $A = \{x \in X : |a(x)| \leq M\}$. Then A has a null complement, and

$$\int_X |af|^2 d\mu = \int_A |af|^2 d\mu \leq M^2 \int_A |f|^2 d\mu < \infty$$

and $af \in L^2(X)$.

It also follows from the previous inequality that $\|L_a(f)\| \leq M\|f\|$, thus L is bounded.

2. Find the norm of this operator

answer We have

$$\|L_a\| = \sup_{f \in L^2, f \neq 0} \frac{\|L_a(f)\|}{\|f\|_2}$$

we showed that for all $f \in L^2(X)$, $\|L_a(f)\| \leq M\|f\|$, where $M = \|a\|_\infty$, thus

$$\|L_a\| \leq M$$

Now Let $\epsilon > 0$, and let $A_\epsilon = \{x : |a(x)| > M - \epsilon\}$. Since $M = \|a\|_\infty$, A_ϵ has positive measure. If $\mu(A_\epsilon) = \infty$, then there exists a subset of A_ϵ with finite positive measure (since the space is σ-finite, there exist a sequence of subsets X_n such that $X = \cup_n X_n$, and $\mu(X_n)$ is finite for all n. Then $0 < \mu(A) \leq \sum_n \mu(A \cap X_n)$, and one of these terms has to be positive by σ-additivity, i.e. there exists n such that $\mu(A \cap X_n) > 0$, and $\mu(A \cap X_n) < \infty$ since X_n has finite measure)

Let $B_\epsilon \subset A_\epsilon$ such that $0 < \mu(B_\epsilon) < \infty$. Finally, consider the function $1_{B_\epsilon} : X \to \mathbb{F}$. We have $1_{B_\epsilon} \in L^2(X)$ since $\int_X |1_{B_\epsilon}|^2 d\mu = \mu(B_\epsilon) < \infty$, and

$$\begin{aligned} \frac{\|L_a(1_{B_\epsilon})\|^2}{\|1_{B_\epsilon}\|^2} &= \frac{1}{\mu(B_\epsilon)} \int_X |a 1_{B_\epsilon}|^2 d\mu \\ &= \frac{1}{\mu(B_\epsilon)} \int_{B_\epsilon} |a|^2 d\mu \\ &> \frac{1}{\mu(B_\epsilon)} \int_{B_\epsilon} (M-\epsilon)^2 d\mu \\ &> (M-\epsilon)^2 \end{aligned}$$

therefore $\|L_a\| \geq M - \epsilon$. Since this holds for arbitrary $\epsilon > 0$, we have

$$\|L_a\| \geq M$$

3. Find the transpose of this operator

answer the transpose is $L_a^* = L_{\bar{a}}$. This operator is in $\mathcal{B}(\mathcal{H}, \mathcal{H})$ for the same reasons as in (a) since $\bar{a} \in L^\infty$. And we have for all f, g

$$\langle L_a(f), g\rangle = \int_X af\bar{g}d\mu = \int_X f\overline{\bar{a}g}d\mu = \langle f, L_{\bar{a}}(g)\rangle$$

which proves the claim

(9.5) Let c_0 denote the space consisting of all complex-valued sequences $a = (a_n, n \in \mathbb{N})$ that satisfy $\lim_{n\to\infty} a_n = 0$. $\|a\|_{c_0} = \max_n |a_n|$. Thus c_0 is simply the space $C_0(\mathbb{N})$, where $\mathbb{N}$ is given the discrete topology. ℓ^∞ is the space of all bounded complex-valued sequences a, with norm $\|a\|_{\ell^\infty} = \sup_n |a_n|$. Obviously c_0 is a proper subset of ℓ^∞.

Show that c_0 has infinite codimension in ℓ^∞. That is, if $S \subset \ell^\infty$ is any finite set, then the smallest closed subspace of ℓ^∞ which contains both c_0 and S, is a proper subspace of ℓ^∞.

proof Let $S = \{s^1, \dots, s^K\}$ be a finite subset of ℓ^∞, and let V be the smallest closed subspace of ℓ^∞ which contains $c_0 \cup S$. Without loss of generality, assume that $s^k \notin c_0$ for all $k \in \{1, \dots, K\}$. Consider the subspace

$$c_0 + \text{span}(s^1, \dots, s^K)$$

we have $\text{span}(s^1, \dots, s^K)$ is closed as a finite dimensional subspace, and c_0 is closed for the following reason: let $a \in \text{cl}(c_0)$, and let (a^m) be a sequence of elements of c_0 that converges to a, i.e.

$$\lim_{m\to\infty} \max_n |a_n^m - a_n| = 0$$

let $\epsilon > 0$. There exists $M \in \mathbb{N}$ such that $\max_n |a_n^M - a_n| \le \epsilon/2$. And since $a^M \in c_0$, there exists $N > 0$ such that for all $n > N$, $|a_n^M| \le \epsilon/2$. Combining the previous bounds, we have for all $n > N$,

$$|a_n| \le |a_n - a_n^M| + |a_n^M| \le \epsilon/2 + \epsilon/2$$

which proves that $a \in c_0$.

Therefore $c_0 + \text{span}(s^1, \dots, s^K)$ is the sum of two closed subspaces, thus it is a closed subspace containing $S \cup c_0$, therefore $V \subset c_0 + \text{span}(s^1, \dots, s^K)$. The reverse inclusion is immediate (any subspace containing c_0 and S must contain $c_0 + \text{span}(s^1, \dots, s^K)$) therefore we have equality.

Next, we show that V is a proper subset of ℓ^∞.

Since $(s_n^1)_n$ is a bounded complex sequence, it has a converging subsequence, $(s^1_{\phi_1(n)})_n$. Now consider the sequence $(s^2_{\phi_1(n)})_n$. This is a subsequence of s^2, and has a converging subsequence $(s^2_{\phi_1\circ\phi_2(n)})_n$. Repeating this construction for the remaining elements of S, we obtain a sequence of extractors $\phi_1, \dots, \phi_K$, such that for all k, $(s^k_{\phi_1\circ\dots\phi_k(n)})_n$ converges. Now let $\phi = \phi_1 \circ \dots \phi_K$. Then

$$(s^k_{\phi(n)})_n$$

is a converging subsequence of s^k for all k. As a consequence, for any element $x \in V = c_0 + \text{span}(s^1, \dots, s^K)$, $(x_{\phi(n)})_n$ is a converging subsequence. Iindeed, there exists $a \in c_0$ and $\lambda_1, \dots, \lambda_K \in \mathbb{C}$ such that $x = a + \sum_{k=1}^K \lambda_k s^k$, then $x_{\phi(n)} = a_{\phi(n)} + \sum_{k=1}^K s^k_{\phi(n)}$, and every term converges.

Not every element of ℓ^∞ satisfies this property. Consider for example the sequence $(y_n)_n$ defined by

$$y_n = \begin{cases} 1 & \text{if } \exists k : n = \phi(2k) \\ 0 & \text{otherwise} \end{cases}$$

then $y \in \ell^\infty$ but $(y_{\phi(n)})_n$ does not converge (even terms equal 1 and odd terms equal 0). Therefore V is a proper subset of ℓ^∞.

(9.6) Let V be a subspace of an inner product space $\mathcal{H}$. Show that the closure of V is equal to $(V^\perp)^\perp$.

proof First we observe that $\mathrm{cl}(V)$ is a subspace since if $x, y \in \mathrm{cl}(V)$, then there exist sequences (x_n) and (y_n) of elements of V such that $x_n \to x$ and $y_n \to y$. Then for all n, $x_n + y_n \in V$ since V is a subspace, and $(x_n + y_n)$ converges to $x + y$, thus $x + y \in V$. Similarly for λx for $\lambda \in \mathbb{F}$.

Next, we show that $V^\perp = \mathrm{cl}(V)^\perp$. We have $V \subset \mathrm{cl}(V)$, thus $\mathrm{cl}(V)^\perp \subset V^\perp$. To show the reverse inclusion, let $y \in V^\perp$. Then for all $x \in V$, $y \perp x$, i.e. the function

$$\begin{aligned} f : X &\to \mathbb{R}_+ \\ x &\mapsto \langle x, y \rangle \end{aligned}$$

is identically zero on V. Since it is continuous (by Cauchy Schwartz), it is also identically zero on $\mathrm{cl}(V)$, which proves that $y \in \mathrm{cl}(V)^\perp$.

To conclude, we have $V^\perp = \mathrm{cl}(V)^\perp$, thus $(V^\perp)^\perp = (\mathrm{cl}(V)^\perp)^\perp$, and since $\mathrm{cl}(V)$ is a closed subspace, $(\mathrm{cl}(V)^\perp)^\perp = \mathrm{cl}(V)$, which proves the claim.

(9.7) Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. Let $L \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, and let L^* be the adjoint of L. Show that

$$\|L^* \circ L\| = \|L \circ L^*\| = \|L\|^2 = \|L^*\|^2$$

Here each occurrence of the symbol $\|\cdot\|$ denotes the norm in $\mathcal{B}(\mathcal{H}_i, \mathcal{H}_j)$ for the appropriate ordered par (i, j).

proof

- we have

$$\begin{aligned} \|L\| &= \sup_{x \in \mathcal{H}_1, \|x\|=1} \|L(x)\| \\ &= \sup_{x \in \mathcal{H}_1, \|x\|=1} \left(\sup_{y \in \mathcal{H}_2, \|y\|=1} \langle L(x), y \rangle \right) && \text{by the Cauchy Shwartz inequality} \\ &= \sup_{x \in \mathcal{H}_1, \|x\|=1, y \in \mathcal{H}_2, \|y\|=1} \langle x, L^*(y) \rangle && \text{by definition of the adjoint} \\ &= \|L^*\| \end{aligned}$$

  which proves $\|L\| = \|L^*\|$

- We have for all $y \in \mathcal{H}_2$ such that $\|y\| = 1$

$$\begin{aligned} \|L(L^*(y))\| &\leq \|L\|\|L^*(y)\| \\ &\leq \|L\|\|L^*\|\|y\| \\ &= \|L^*\|^2 \end{aligned}$$

  therefore taking the sup over all such y, we have

$$\|L \circ L^*\| = \sup_{y \in \mathcal{H}_2, \|y\|=1} \|L(L^*(y)) \leq \|L^*\|^2$$

  to prove the reverse inequality, we have for all $y \in \mathcal{H}_2$ such that $\|y\| = 1$

$$\begin{aligned} \|L^*(y)\|^2 &= \langle L^*(y), L^*(y) \rangle \\ &= \langle y, L \circ L^*(y) \rangle \\ &\leq \|L \circ L^*(y)\| && \text{by Cauchy Schwartz} \end{aligned}$$

  then taking the sup over all such y, we have $\|L^*\|^2 \leq \|L \circ L^*\|$. This proves $\|L \circ L^*\| = \|L^*\|^2$

- by applying the previous equality to the operator $L^{**} \circ L^*$, we have

$$\|L^{**} \circ L^*\| = \|L^{**}\|^2$$

finally, since for all $x \in \mathcal{H}_2$ and for all $y \in \mathcal{H}_1$

$$\langle x, L^{**}(y)\rangle = \langle L^*(x), y\rangle = \langle x, L(y)\rangle$$

we have $L^{**} = L$, which concludes the proof.

(9.8) Let $\mathcal{H}$ be an infinite dimensional Hilbert space over $\mathbb{C}$. Show that there exists a linear functional $\ell : \mathcal{H} \to \mathbb{C}$ which is not bounded.

proof Let S be the set of unbounded linear operators $L : V \to \mathbb{C}$ defined on a subspace $V \subset \mathcal{H}$.

First, we show that S is non empty: let $\{e_n, n \in \mathbb{N}\}$ be a countable orthonormal set (such a set exists since the space is supposed to be infinite dimensional), and consider the subspace V of vectors that are a linear combination of finitely many e_n. Define the linear operator

$$\begin{aligned} L : V &\to \mathbb{C} \\ v &\mapsto L(v) = \sum_{n=1}^{\infty} n\langle v, e_n\rangle \end{aligned}$$

the sum only has finitely many non-zero terms, by definition of the subspace V. This operator is unbounded: for all $n \in \mathbb{N}$, $L(e_n) = n$, thus $\frac{L(e_n)|}{\|e_n\|} \geq n$ for all n.

Now equip S with the partial order $L \leq U$ if and only if $G_L \subset G_U$, where $G_L = \{(v, L(v), v \in V\}$ is the graph of L. Any chain in S has an upper bound in S: let $(L_\alpha)_{\alpha \in A}$ be such a chain, where $L_\alpha : V_\alpha \to \mathbb{C}$, and the graphs of L_α are nested. Consider the function L such that $G_L = \cup_{\alpha \in A} G_{L_\alpha}$ (the union indeed defines a graph since G_{L_α} are nested). The domain of L is $V = \cup_{\alpha \in A} V_\alpha$, and is a subspace (union of subspaces). We have

- L is linear: if $v_1, v_2 \in V$, then there exist $\alpha_1, \alpha_2 \in A$ such that $v_1 \in V_{\alpha_1}$ and $v_2 \in V_{\alpha_2}$. Since we have a chain, assume without loss of generality that $G_{L_{\alpha_1}} \subset G_{L_{\alpha_2}}$. Then $V_1 \subset V_2$, and

$$\begin{aligned} L(v_1 + v_2) &= L_{\alpha_2}(v_1 + v_2) \\ &= L_{\alpha_2}(v_1) + L_{\alpha_2}(v_2) && \text{by linearity of } L_{\alpha_2} \\ &= L(v_1) + L(v_2) \end{aligned}$$

- L is unbounded, since it extends all L_α in the chain, and any such L_α is unbounded.

By Zorn's lemma, S has a maximal element $L_0 : V_0 \to \mathbb{C}$. Then $V_0 = \mathcal{H}$, for it were not, then there would exist $x_1 \notin V_0$, then consider the subspace $\tilde{V_0} = V_0 \oplus \operatorname{span}(x_1)$, and define

$$\begin{aligned} \tilde{L}_0 : \tilde{V_0} &\to \mathbb{C} \\ v &\mapsto L_0(\pi(v)) \end{aligned}$$

where π is the projection on V_0 in the direction x_1. We have

- $\tilde{L}_0$ is linear since it is the composition of two linear functions.
- $\tilde{L}_0$ extends L_0, since π restricted to V_0 is the identity. In particular $\tilde{L}_0$ is unbounded, and is an element of S

this contradicts maximality of L_0, since the graph of $\tilde{L}_0$ is a proper subset of the graph of L_0. Therefore L_0 is defined on $\mathcal{H}$, which concludes the construction.

(9.9) Let $p \in (2, \infty)$. Let V be a closed subspace of $L^p = L^p(X, \mathcal{A}, \mu)$. Show that for any $f \in L^p$, there exists $g \in V$ satisfying $\|f - g\|_p = \inf_{h \in V} \|f - h\|_p$. Show that g is unique.

proof Let $p \in (2, \infty)$. Then we have from problem (3.2), for all $f, g \in L^p$

$$\|f + g\|_p^p + \|f - g\|_p^p \leq 2^{p-1}(\|f\|_p^p + \|g\|_p^p)$$

let $d = \inf_{h \in V} \|f - h\|_p$. Then for all $n \in \mathbb{N}$, there exists $h_n \in V$ such that $\|h_n - f\|_p \leq d - 1/n$. Using the above inequality, we have for all m, n

$$\|(h_m - f) + (h_n - f)\|_p^p + \|(h_m - f) - (h_n - f)\|_p^p \leq 2^{p-1}(\|h_m - f\|_p^p + \|h_n - f\|_p^p)$$

i.e.

$$\|h_m - h_n\|_p^p \leq 2^{p-1}(\|h_m - f\|_p^p + \|h_n - f\|_p^p) - 2^p \|\frac{h_m + h_n}{2} - f\|_p^p \tag{1}$$

finally, using the fact that $\frac{h_m + h_n}{2} \in V$, we have $\|\frac{h_m + h_n}{2} - f\|_p \geq d$, therefore

$$\|h_m - h_n\|_p^p \leq 2^{p-1}\left(\left(d + \frac{1}{n}\right)^p + \left(d + \frac{1}{m})^p\right)\right) - 2^p d^p$$

which converges to 0 as m and n tend to infinity. Thus (h_n) is a Cauchy sequence in V. Since L^p is complete, (h_n) converges, and since V is closed, its limit $\lim_n h_n \in V$. This proves existence.

To show uniqueness, let $g_1, g_2 \in V$ such that $\|f - g_1\|_p = \|f - g_2\|_p = d$. Then using the same bound (1), we have

$$\begin{aligned} \|g_1 - g_2\|_p^p &\leq 2^{p-1}(\|f - g_1\|_p^p + \|f - g_2\|_p^p) - 2^p \|\frac{g_1 + g_2}{2} - f\|_p^p \\ &= 2^{p-1}(2d^p) - 2^p \|\frac{g_1 + g_2}{2} - f\|_p^p \end{aligned}$$

and using the fact that $\|\frac{g_1 + g_2}{2} - f\|_p \geq d$,

$$\|g_1 - g_2\|_p^p \leq 2^{p-1}(2d^p) - 2^p d^p = 0$$

therefore $g_1 = g_2$