(9.1) Let \((H, \langle \cdot, \cdot \rangle)\) be an inner product space. Show that if two nonzero vectors \(x, y \in H\) satisfy \(|\langle x, y \rangle| = \|x\|\|y\|\), then there exists \(c \in \mathbb{F}\) such that \(y = cx\).

**proof** We have for all \(c \in \mathbb{C}\),
\[
\|y - cx\|^2 = \langle y - cx, y - cx \rangle \\
= \|y\|^2 + \|cx\|^2 - \langle y, cx \rangle - \langle cx, y \rangle \\
= \|y\|^2 + |c|^2\|x\|^2 - 2\Re(c\langle x, y \rangle)
\]
consider in particular \(c = \frac{\langle x, y \rangle}{\|x\|^2}\) (well defined since \(x\) is a nonzero vectors). Then
\[
\|y - cx\|^2 = \|y\|^2 + \frac{|\langle x, y \rangle|^2}{\|x\|^4}\|x\|^2 - 2\Re\left(\frac{\langle x, y \rangle}{\|x\|^2}\langle x, y \rangle\right) \\
= \|y\|^2 + \frac{|\langle x, y \rangle|^2}{\|x\|^2} - 2\frac{|\langle x, y \rangle|^2}{\|x\|^2} \\
= \|y\|^2 - \frac{|\langle x, y \rangle|^2}{\|x\|^2} \\
= 0
\]
which proves that \(y = cx\).

(9.2) Let \((X, \mathcal{A}, \mu)\) be a measure space. Let \(\phi_n, n \in \mathbb{N}\) be an orthonormal basis for \(L^2(X) = L^2(X, \mathcal{A}, \mu)\). This is a space of equivalence classes of complex-valued measurable functions. Define functions
\[
\phi_m \otimes \phi_n : X \times X \to \mathbb{C} \\
(x, y) \mapsto \phi_m(x)\phi_n(y)
\]
Show that \(\mathcal{O} = \{\phi_m \otimes \phi_n, (m, n) \in \mathbb{N}^2\}\) is an orthonormal basis for \(L^2(X \times X, \mathcal{A} \times \mathcal{A}, \mu \times \mu)\).

**proof** We show that \(\mathcal{O}\) is an orthonormal set of \(L^2(X \times X)\) functions, and that for all \(f \in L^2(X \times X)\),
\[
\|f\|^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle f, \phi_m \otimes \phi_n \rangle|^2.
\]
We have
• for all $m, n$, we have

$$\|\phi_n \otimes \phi_m\|^2 = \int_{X \times X} |\phi_n(x)\phi_m(y)|^2 d\mu_1 \times \mu_2$$

$$= \int_X \left( \int_X |\phi_n(x)|^2 |\phi_m(y)|^2 d\mu_2(y) \right) d\mu_1(x)$$

$$= \|\phi_n\|^2 \|\phi_m\|^2$$

which is finite since $\phi_n, \phi_m \in L^2(X)$. Thus $\phi_n \otimes \phi_m \in L^2(X \times X)$.

• we have for all $m, n, i, j$

$$\langle \phi_m \otimes \phi_n, \phi_i \otimes \phi_j \rangle = \int_{X \times X} \phi_n(x)\phi_m(y)\phi_i(x)\phi_j(y) d\mu_1 \times \mu_2(x, y)$$

$$= \int_X \left( \int_X \phi_n(x)\phi_i(x)\phi_m(y)\phi_j(y) d\mu_2(y) \right) d\mu_1(x)$$

$$= \int_X \phi_n(x)\phi_i(x) \int_X \phi_m(y)\phi_j(y) d\mu_2(y)$$

$$= \langle \phi_m, \phi_i \rangle \langle \phi_n, \phi_j \rangle$$

$$= \begin{cases} 1 & \text{if } (m, n) = (i, j) \\ 0 & \text{otherwise} \end{cases}$$

thus $\mathcal{O}$ is an orthonormal set.

• now let $f \in L^2(X \times X)$. We have

$$\langle f, \phi_n \otimes \phi_m \rangle = \int_{X \times X} f(x, y)\overline{\phi_n(x)\phi_m(y)} d\mu_1 \times \mu_2(x, y)$$

$$= \int_X \left( \int_X f(y)\overline{\phi_m(y)} d\mu_2(y) \right) \overline{\phi_n(x)} d\mu_1(x)$$

$$= \int_X \langle f \overline{x}, \phi_m \rangle \overline{\phi_n(x)} d\mu_1(x)$$

$$= \langle \pi_m, \phi_n \rangle$$

where $\pi_m : x \mapsto \langle f \overline{x}, \phi_n \rangle$. Next, we have

$$\|f\|^2_{L^2(X \times X)} = \int_{X \times X} |f(x, y)|^2 d\mu_1 \times \mu_2$$

$$= \int_X \left( \int_X |f(x)|^2 d\mu_2(y) \right) d\mu_1(x)$$

$$= \int_X \|f\|^2 \mu_1(x)$$

and in particular, $f^x \in L^2(X)$ for a.e. $x$ (otherwise $\|f\|^2 = \infty$). Let $A$ be the set of $x$ such that $f^x \in L^2$. For all $x \in A$, since $\{\phi_n, n \in \mathbb{N}\}$ is an orthonormal basis of $L^2(X)$, we have

$$\|f^x\|^2 = \sum_{m=1}^{\infty} |\langle f^x, \phi_m \rangle|^2 = \sum_{m=1}^{\infty} |\pi_m(x)|^2$$
therefore
\[ \|f\|^2 = \int_A \sum_{m=1}^{\infty} |\pi_m(x)|^2 d\mu_1(x) \]
\[ = \sum_{m=1}^{\infty} \int_A |\pi_m(x)|^2 d\mu_1(x) \quad \text{by the DCT} \]
\[ = \sum_{m=1}^{\infty} \|\pi_m\|^2 \]

here each \( \pi_m \) is in \( L^2(X) \) (the sum is finite), thus \( \|\pi_m\|^2 = \sum_{n=1}^{\infty} \langle \pi_m, \phi_n \rangle \). Combining this with the last equality, we have
\[ \|f\|^2 = \sum_{m,n \in \mathbb{N}} \langle \pi_m, \phi_n \rangle \]
\[ = \sum_{m,n \in \mathbb{N}} \langle \pi_m, \phi_n \rangle \quad \text{by Fubini’s theorem} \]
\[ = \sum_{m,n \in \mathbb{N}} \langle f, \phi_n \otimes \phi_m \rangle \]

as claimed

\[ (9.3) \quad \text{Let } \mathcal{H} \text{ be a Hilbert space, and let } L \in \mathcal{B} (\mathcal{H}, \mathcal{H}) \text{ be an isometry in the sense that } \|L(x)\| = \|x\| \text{ for every } x \in \mathcal{H}. \]

1. Prove that the range of \( L \) is a closed subspace of \( \mathcal{H} \).

**proof** The range of \( L \) is a subspace since \( L \) is linear (if \( y_1, y_2 \in L(\mathcal{H}) \), then there exist \( x_1, x_2 \in \mathcal{H} \) such that \( L(x_1) = y_1 \) and \( L(x_2) = y_2 \). Then \( y_1 + y_2 = L(x_1 + x_2) \in L(\mathcal{H}) \), and for all \( \lambda \in \mathbb{F} \), \( \lambda y = L(\lambda x) \in L(\mathcal{H}) \)).

Now let \( y \) be a vector in the closure of \( L(\mathcal{H}) \). Then there exists a sequence \( (y_n)_n \) of elements of \( L(\mathcal{H}) \) that converges to \( y \). For all \( n \), there exists \( x_n \in \mathcal{H} \) such that \( y_n = L(x_n) \). Then \( (x_n) \) is de Cauchy since
\[ \|x_n - x_m\| = \|L(x_n - x_m)\| = \|L(x_n) - L(x_m)\| = \|y_n - y_m\| \]
and \( (y_n) \) is Cauchy. Since the space is Hilbert, \( (x_n) \) converges. Let \( x \) be its limit. Then we have
\[ \|L(x) - L(x_n)\| = \|x - x_n\| \]
which converges to zero as \( n \to \infty \), thus \( L(x) = \lim_n L(x_n) = \lim_n y_n = y \). Therefore \( L(x) = y \) and \( y \in L(\mathcal{H}) \).

2. Give an example of an isometry \( L \) which is not surjective.

**answer** Consider the space \( \ell^2 \) of complex sequences \( (a_n) \) such that \( \sum_{n \in \mathbb{N}} |a_n|^2 < \infty \), and consider the linear operator that shifts the sequence to the right
\[
L : \ell^2 \to \ell^2 \\
(a_1, a_2, \ldots) \mapsto (0, a_1, a_2, \ldots)
\]

\( L \) is clearly linear, an isometry, and the sequence \( (1, 0, 0, \ldots) \) is not in the range of \( L \).
Let \((X, A, \mu)\) be a measure space. Let \(H = L^2(X, A, \mu)\), equivalence classes of complex-valued functions.

1. Show that for any function \(a \in L^\infty(X)\),
   \[
   L_a : H \to H
   \]
   \[
   f \mapsto L_a(f) = x \mapsto a(x)f(x)
   
   \]
defines an operator in \(B(H)\).

   **proof** \(L_a\) is clearly linear, since
   \[
   L_a(f + g) = a(f + g) = af + ag = L_a(f) + L_a(g)
   \]
   \[
   L(\lambda f) = a\lambda f = \lambda L_a(f)
   
   \]
   We verify that for all \(a \in L^\infty(X)\) and all \(f \in L^2(X)\), \(af\) is indeed in \(L^2(X)\). Let
   \[
   M = \|a\|_\infty = \inf\{N > 0 : \mu(\{x : |a(x)| > N\}) = 0\}
   \]
   be the essential supremum of \(a\). Then for a.e. \(x\), \(|a(x)| \leq M\). Let \(A = \{x \in X : |a(x)| \leq M\}\). Then \(A\) has a null complement, and
   \[
   \int_X |af|^2d\mu = \int_A |af|^2d\mu \leq M^2 \int_A |f|^2d\mu < \infty
   
   \]
   and \(af \in L^2(X)\).
   
   It also follows from the previous inequality that \(\|L_a(f)\| \leq M\|f\|\), thus \(L\) is bounded.

2. Find the norm of this operator

   **answer** We have
   \[
   \|L_a\| = \sup_{f \in L^2, f \neq 0} \frac{\|L_a(f)\|}{\|f\|_2}
   \]
   we showed that for all \(f \in L^2(X)\), \(\|L_a(f)\| \leq M\|f\|\), where \(M = \|a\|_\infty\), thus
   \[
   \|L_a\| \leq M
   
   \]
   Now let \(\epsilon > 0\), and let \(A_\epsilon = \{x : |a(x)| > M - \epsilon\}\). Since \(M = \|a\|_\infty\), \(A_\epsilon\) has positive measure. If \(\mu(A_\epsilon) = \infty\), then there exists a subset of \(A_\epsilon\) with finite positive measure (since the space is \(\sigma\)-finite, there exist a sequence of subsets \(X_n\) such that \(X = \bigcup_n X_n\), and \(\mu(X_n)\) is finite for all \(n\). Then
   \[
   0 < \mu(A) \leq \sum_n \mu(A \cap X_n),
   \]
   and one of these terms has to be positive by \(\sigma\)-additivity, i.e. there exists \(n\) such that \(\mu(A \cap X_n) > 0\), and \(\mu(A \cap X_n) < \infty\) since \(X_n\) has finite measure)
   
   Let \(B_\epsilon \subset A_\epsilon\) such that \(0 < \mu(B_\epsilon) < \infty\). Finally, consider the function \(1_{B_\epsilon} : X \to \mathbb{F}\). We have \(1_{B_\epsilon} \in L^2(X)\) since
   \[
   \int_X |1_{B_\epsilon}|^2d\mu = \mu(B_\epsilon) < \infty
   \]
   and
   \[
   \frac{\|L_a(1_{B_\epsilon})\|_2^2}{\|1_{B_\epsilon}\|_2^2} = \frac{1}{\mu(B_\epsilon)} \int_{B_\epsilon} |a1_{B_\epsilon}|^2d\mu
   \]
   \[
   = \frac{1}{\mu(B_\epsilon)} \int_{B_\epsilon} |a|^2d\mu
   \]
   \[
   > \frac{1}{\mu(B_\epsilon)} \int_{B_\epsilon} (M - \epsilon)^2d\mu
   \]
   \[
   > (M - \epsilon)^2
   
   \]
   therefore \(\|L_a\| \geq M - \epsilon\). Since this holds for arbitrary \(\epsilon > 0\), we have
   \[
   \|L_a\| \geq M
   
   \]

3. Find the transpose of this operator
Let \( \text{subspace } S \) and which proves that for all \( n > N \) that for all \( c \in S \) which contains \( c \) (9.5) Let \( c_0 \) denote the space consisting of all complex-valued sequences \( a = (a_n, n \in \mathbb{N}) \) that satisfy \( \lim_{n \to \infty} a_n = 0. \) Thus \( c_0 \) is simply the space \( C_0(\mathbb{N}) \), where \( \mathbb{N} \) is given the discrete topology. \( \ell^\infty \) is the space of all bounded complex-valued sequences \( a \), with norm \( \|a\|_{\ell^\infty} = \sup_n |a_n| \). Obviously \( c_0 \) is a proper subset of \( \ell^\infty \).

Show that \( c_0 \) has infinite codimension in \( \ell^\infty \). That is, if \( S \subset \ell^\infty \) is any finite set, then the smallest closed subspace of \( \ell^\infty \) which contains both \( c_0 \) and \( S \), is a proper subspace of \( \ell^\infty \).

**proof** Let \( S = \{s^1, \ldots, s^K\} \) be a finite subset of \( \ell^\infty \), and let \( V \) be the smallest closed subspace of \( \ell^\infty \) which contains \( c_0 \cup S \). Without loss of generality, assume that \( s^k \notin c_0 \) for all \( k \in \{1, \ldots, K\} \). Consider the subspace

\[
c_0 + \text{span}(s^1, \ldots, s^K)
\]

we have \( \text{span}(s^1, \ldots, s^K) \) is closed as a finite dimensional subspace, and \( c_0 \) is closed for the following reason: let \( a \in c_1(c_0) \), and let \( (a^m) \) be a sequence of elements of \( c_0 \) that converges to \( a \), i.e.

\[
\lim_{m \to \infty} \max_n |a^m_n - a_n| = 0
\]

let \( \epsilon > 0 \). There exists \( M \in \mathbb{N} \) such that \( \max_n |a^M_n - a_n| \leq \epsilon/2 \). And since \( a^M \in c_0 \), there exists \( N > 0 \) such that for all \( n > N \), \( |a^M_n| \leq \epsilon/2 \). Combining the previous bounds, we have for all \( n > N \),

\[
|a_n| \leq |a^M_n - a_n| + |a^M_n| \leq \epsilon/2 + \epsilon/2
\]

which proves that \( a \in c_0 \).

Therefore \( c_0 + \text{span}(s^1, \ldots, s^K) \) is the sum of two closed subspaces, thus it is a closed subspace containing \( S \cup c_0 \), therefore \( V \subset c_0 + \text{span}(s^1, \ldots, s^K) \). The reverse inclusion is immediate (any subspace containing \( c_0 \) and \( S \) must contain \( c_0 + \text{span}(s^1, \ldots, s^K) \)) therefore we have equality.

Next, we show that \( V \) is a proper subset of \( \ell^\infty \).

Since \( (s^s_n) \) is a bounded complex sequence, it has a converging subsequence, \( (s^s_{\phi_1(n)}) \). Now consider the sequence \( (s^2_{\phi_2(n)}) \). This is a subsequence of \( s^2 \), and has a converging subsequence \( (s^2_{\phi_{1,0}(n)}) \). Repeating this construction for the remaining elements of \( S \), we obtain a sequence of extractors \( \phi_1, \ldots, \phi_K \), such that for all \( k \), \( (s^k_{\phi_1, \ldots, \phi_k(n)}) \) converges. Now let \( \phi = \phi_1 \circ \ldots \circ \phi_K \). Then

\[
(s^k_{\phi(n)})
\]

is a converging subsequence of \( s^k \) for all \( k \). As a consequence, for any element \( x \in V = c_0 + \text{span}(s^1, \ldots, s^K), \)

\[
(x_{\phi(n)})
\]

is a converging subsequence. Indeed, there exists \( a \in c_0 \) and \( \lambda_1, \ldots, \lambda_K \in \mathbb{C} \) such that \( x = a + \sum_{k=1}^{K} \lambda_k s^k \), then \( x_{\phi(n)} = a_{\phi(n)} + \sum_{k=1}^{K} s^k_{\phi(n)} \), and every term converges.

Not every element of \( \ell^\infty \) satisfies this property. Consider for example the sequence \( (y_n) \) defined by

\[
y_n = \begin{cases} 1 & \text{if } \exists k : n = \phi(2k) \\ 0 & \text{otherwise} \end{cases}
\]

then \( y \in \ell^\infty \) but \( (y_{\phi(n)}) \) does not converge (even terms equal 1 and odd terms equal 0). Therefore \( V \) is a proper subset of \( \ell^\infty \).
(9.6) Let \( V \) be a subspace of an inner product space \( \mathcal{H} \). Show that the closure of \( V \) is equal to \( (V^\perp)^\perp \).

**proof** First we observe that \( \text{cl}(V) \) is a subspace since if \( x, y \in \text{cl}(V) \), then there exist sequences \((x_n)\) and \((y_n)\) of elements of \( V \) such that \( x_n \to x \) and \( y_n \to y \). Then for all \( n \), \( x_n + y_n \in V \) since \( V \) is a subspace, and \( (x_n + y_n) \) converges to \( x + y \), thus \( x + y \in V \). Similarly for \( \lambda x \) for \( \lambda \in \mathbb{F} \).

Next, we show that \( V^\perp = \text{cl}(V)^\perp \). We have \( V \subset \text{cl}(V) \), thus \( \text{cl}(V)^\perp \subset V^\perp \). To show the reverse inclusion, let \( y \in V^\perp \). Then for all \( x \in V, y \perp x \), i.e. the function

\[
f : X \to \mathbb{R}_+
\]

\[
x \mapsto \langle x, y \rangle
\]

is identically zero on \( V \). Since it is continuous (by Cauchy Schwartz), it is also identically zero on \( \text{cl}(V) \), which proves that \( y \in \text{cl}(V)^\perp \).

To conclude, we have \( V^\perp = \text{cl}(V)^\perp \), thus \( (V^\perp)^\perp = (\text{cl}(V)^\perp)^\perp \), and since \( \text{cl}(V) \) is a closed subspace, \( (\text{cl}(V)^\perp)^\perp = \text{cl}(V) \), which proves the claim.

(9.7) Let \( \mathcal{H}_1, \mathcal{H}_2 \) be Hilbert spaces. Let \( L \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \), and let \( L^* \) be the adjoint of \( L \). Show that

\[
\|L^* \circ L\| = \|L \circ L^*\| = \|L\|^2 = \|L^*\|^2
\]

Here each occurrence of the symbol \( \| \cdot \| \) denotes the norm in \( \mathcal{B}(\mathcal{H}_i, \mathcal{H}_j) \) for the appropriate ordered par \((i, j)\).

**proof**

- we have

\[
\|L\| = \sup_{x \in \mathcal{H}_1, \|x\| = 1} \|L(x)\|
\]

\[
= \sup_{x \in \mathcal{H}_1, \|x\| = 1} \left( \sup_{y \in \mathcal{H}_2, \|y\| = 1} \langle L(x), y \rangle \right)
\]

by the Cauchy Schwartz inequality

\[
= \sup_{x \in \mathcal{H}_1, \|x\| = 1} \langle x, L^*(y) \rangle
\]

by definition of the adjoint

\[
= \|L^*\|
\]

which proves \( \|L\| = \|L^*\| \)

- We have for all \( y \in \mathcal{H}_2 \) such that \( \|y\| = 1 \)

\[
\|L(L^*(y))\| \leq \|L\|\|L^*(y)\|
\]

\[
\leq \|L\|\|L^*\|\|y\|
\]

\[
= \|L^*\|^2
\]

therefore taking the sup over all such \( y \), we have

\[
\|L \circ L^*\| = \sup_{y \in \mathcal{H}_2, \|y\| = 1} \|L(L^*(y))\| \leq \|L^*\|^2
\]

to prove the reverse inequality, we have for all \( y \in \mathcal{H}_2 \) such that \( \|y\| = 1 \)

\[
\|L^*(y)\|^2 = \langle L^*(y), L^*(y) \rangle
\]

\[
= \langle y, L \circ L^*(y) \rangle
\]

\[
\leq \|L \circ L^*(y)\|
\]

by Cauchy Schwartz

then taking the sup over all such \( y \), we have \( \|L^*\|^2 \leq \|L \circ L^*\| \). This proves \( \|L \circ L^*\| = \|L^*\|^2 \)
• by applying the previous equality to the operator $L^{**} \circ L^*$, we have

$$\|L^{**} \circ L^*\| = \|L^{**}\|^2$$

finally, since for all $x \in \mathcal{H}_2$ and for all $y \in \mathcal{H}_1$

$$\langle x, L^{**}(y) \rangle = \langle L^*(x), y \rangle = \langle x, L(y) \rangle$$

we have $L^{**} = L$, which concludes the proof.

(9.8) Let $\mathcal{H}$ be an infinite dimensional Hilbert space over $\mathbb{C}$. Show that there exists a linear functional $\ell : \mathcal{H} \to \mathbb{C}$ which is not bounded.

**proof** Let $S$ be the set of unbounded linear operators $L : V \to \mathbb{C}$ defined on a subspace $V \subset \mathcal{H}$.

First, we show that $S$ is non empty: let $\{e_n, n \in \mathbb{N}\}$ be a countable orthonormal set (such a set exists since the space is supposed to be infinite dimensional), and consider the subspace $V$ of vectors that are a linear combination of finitely many $e_n$. Define the linear operator

$$L : V \to \mathbb{C}$$

$$v \mapsto L(v) = \sum_{n=1}^{\infty} n\langle v, e_n \rangle$$

the sum only has finitely many non-zero terms, by definition of the subspace $V$. This operator is unbounded: for all $n \in \mathbb{N}$, $L(e_n) = n$, thus $\frac{L(e_n)}{n} \geq n$ for all $n$.

Now equip $S$ with the partial order $L \leq U$ if and only if $G_L \subset G_U$, where $G_L = \{(v, L(v), v \in V)\}$ is the graph of $L$. Any chain in $S$ has an upper bound in $S$: let $(L_\alpha)_{\alpha \in A}$ be such a chain, where $L_\alpha : V_\alpha \to \mathbb{C}$, and the graphs of $L_\alpha$ are nested. Consider the function $L$ such that $G_L = \bigcup_{\alpha \in A} G_{L_\alpha}$ (the union indeed defines a graph since $G_{L_\alpha}$ are nested). The domain of $L$ is $V = \bigcup_{\alpha \in A} V_\alpha$, and is a subspace (union of subspaces). We have

• $L$ is linear: if $v_1, v_2 \in V$, then there exist $\alpha_1, \alpha_2 \in A$ such that $v_1 \in V_{\alpha_1}$ and $v_2 \in V_{\alpha_2}$. Since we have a chain, assume without loss of generality that $G_{L_{\alpha_1}} \subset G_{L_{\alpha_2}}$. Then $V_1 \subset V_2$, and

$$L(v_1 + v_2) = L_{\alpha_2}(v_1 + v_2)$$

$$= L_{\alpha_2}(v_1) + L_{\alpha_2}(v_2)$$

by linearity of $L_{\alpha_2}$

$$= L(v_1) + L(v_2)$$

• $L$ is unbounded, since it extends all $L_\alpha$ in the chain, and any such $L_\alpha$ is unbounded.

By Zorn’s lemma, $S$ has a maximal element $L_0 : V_0 \to \mathbb{C}$. Then $V_0 = \mathcal{H}$, for it were not, then there would exist $x_1 \notin V_0$, then consider the subspace $\tilde{V}_0 = V_0 \oplus \text{span}(x_1)$, and define

$$\tilde{L}_0 : \tilde{V}_0 \to \mathbb{C}$$

$$v \mapsto L_0(\pi(v))$$

where $\pi$ is the projection on $V_0$ in the direction $x_1$. We have

• $\tilde{L}_0$ is linear since it is the composition of two linear functions.

• $\tilde{L}_0$ extends $L_0$, since $\pi$ restricted to $V_0$ is the identity. In particular $\tilde{L}_0$ is unbounded, and is an element of $S$ this contradicts maximality of $L_0$, since the graph of $\tilde{L}_0$ is a proper subset of the graph of $L_0$. Therefore $L_0$ is defined on $\mathcal{H}$, which concludes the construction.
(9.9) Let $p \in (2, \infty)$. Let $V$ be a closed subspace of $L^p = L^p(X, \mathcal{A}, \mu)$. Show that for any $f \in L^p$, there exists $g \in V$ satisfying $\|f - g\|_p = \inf_{h \in V} \|f - h\|_p$. Show that $g$ is unique.

**proof** Let $p \in (2, \infty)$. Then we have from problem (3.2), for all $f, g \in L^p$

$$\|f + g\|_p^p + \|f - g\|_p^p \leq 2^{p-1}(\|f\|_p^p + \|g\|_p^p)$$

let $d = \inf_{h \in V} \|f - h\|_p$. Then for all $n \in \mathbb{N}$, there exists $h_n \in V$ such that $\|h_n - f\|_p \leq d - 1/n$. Using the above inequality, we have for all $m, n$

$$\|(h_m - f) + (h_n - f)\|_p^p + \|(h_m - f) - (h_n - f)\|_p^p \leq 2^{p-1}(\|h_m - f\|_p^p + \|h_n - f\|_p^p)$$

i.e.

$$\|h_m - h_n\|_p^p \leq 2^{p-1}(\|h_m - f\|_p^p + \|h_n - f\|_p^p) - 2^p \frac{h_m + h_n}{2} - f\|_p^p$$

(1)

finally, using the fact that $\frac{h_m + h_n}{2} \in V$, we have $\|\frac{h_m + h_n}{2} - f\|_p \geq d$, therefore

$$\|h_m - h_n\|_p^p \leq 2^{p-1} \left( \left( d + \frac{1}{n} \right)^p + \left( d + \frac{1}{m} \right)^p \right) - 2^p d^p$$

which converges to 0 as $m$ and $n$ tend to infinity. Thus $(h_n)$ is a Cauchy sequence in $V$. Since $L^p$ is complete, $(h_n)$ converges, and since $V$ is closed, its limit $\lim_{n} h_n \in V$. This proves existence.

To show uniqueness, let $g_1, g_2 \in V$ such that $\|f - g_1\|_p = \|f - g_2\|_p = d$. Then using the same bound (1), we have

$$\|g_1 - g_2\|_p^p \leq 2^{p-1}(\|f - g_1\|_p^p + \|f - g_2\|_p^p) - 2^p \frac{g_1 + g_2}{2} - f\|_p^p$$

$$= 2^{p-1}(2d^p) - 2^p \frac{g_1 + g_2}{2} - f\|_p^p$$

and using the fact that $\frac{g_1 + g_2}{2} - f\|_p \geq d$,

$$\|g_1 - g_2\|_p^p \leq 2^{p-1}(2d^p) - 2^p d^p = 0$$

therefore $g_1 = g_2$