MATH 202B - Problem Set 8

Walid Krichene (23265217)

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(8.1) Let μ be a complex measure on $(X, \mathcal{A})$. The associated positive measure $|\mu|$ is defined by, for all $E \in \mathcal{A}$

$$|\mu|(E) = \sup \sum_j |\mu(E_j)|$$

where the sup is taken over all decompositions of E as a union of countably many pairwise disjoint measurable subsets E_j. Show that there exists a measurable function h satisfying $|h| = 1$ $|\mu|$-almost everywhere, such that $\mu = h|\mu|$, that is for all $E \in \mathcal{A}$,

$$\mu(E) = \int_E h d|\mu|$$

proof In order to apply the Radon-Nykodym theorem, we first show that $|\mu|(X)$ is finite. Decompose the complex measure μ into

$$\mu = (\mu_R^+ - \mu_R^-) + i(\mu_I^+ - \mu_I^-)$$

where all measures in the decomposition are (positive) finite Radon measures. Then we have for all measurable E, using the fact that for any complex $z = a + ib$, $|z| \leq |a| + |b|$

$$|\mu(E)| \leq |\mu_R(E)| + |\mu_I(E)| = \mu_R^+(E) + \mu_R^-(E) + \mu_I^+(E) + \mu_I^-(E)$$

therefore for any partition $X = \sqcup_i E_i$,

$$\begin{aligned}\sum_i |\mu(E_i)| &\leq \sum_i \left(\mu_R^+(E_i) + \mu_R^-(E_i) + \mu_I^+(E_i) + \mu_I^-(E_i)\right) \\ &= \mu_R^+(X) + \mu_R^-(X) + \mu_I^+(X) + \mu_I^-(X)\end{aligned}$$

which is finite. Taking the sup over all such partitions, we obtain $|\mu|(X)$ is finite.

Next, we have by definition of $|\mu|$, for all measurable E,

$$|\mu|(E) \geq |\mu(E)| \tag{1}$$

since the sup in the definition of $|\mu|$ is taken over all partitions, in particular the partition that contains a single element $\{E\}$.

We have in particular $\mu << |\mu|$, and by the Radon-Nikodym Theorem, there exists a measurable function h such that for all measurable E

$$\mu(E) = \int_E h d|\mu|$$

- first, we show that $|h(x)| \leq 1$ for $|\mu|$-a.e. $x \in X$. Let $A = \{z \in \mathbb{C} : |z| > 1\}$. We seek to show that $h^{-1}(A)$ is a null set. Since A is an open subset of $\mathbb{C}$, it is the union of countably many closed balls

$$A = \cup_{n \in \mathbb{N}} D(z_n, r_n)$$

let $E_n = h^{-1}(D(z_n, r_n))$. E_n is measurable, and a null set, for otherwise we would have

$$\begin{aligned}
\left| \frac{\int_{E_n} hd|\mu|}{|\mu|(E_n)} - z_n \right| &= \left| \frac{\int_{E_n} hd|\mu|}{|\mu|(E_n)} - z_n \right| \\
&= \frac{1}{|\mu|(E_n)} \left| \int_{E_n} (h - z_n) d|\mu| \right| \\
&\leq \frac{1}{|\mu|(E_n)} \int_{E_n} |h - z_n| d\mu \\
&\leq \frac{1}{|\mu|(E_n)} \int_{E_n} r_n d\mu \qquad \text{since } h(E_n) = D(z_n, r_n) \\
&= r_n
\end{aligned}$$

and it would follow that $\frac{\mu(E_n)}{|\mu|(E_n)} \in D(z_n, r_n) \subset A$ which contradicts the fact $|\mu(E_n)| \leq |\mu|(E_n)$ in (1). Therefore $|\mu|(E_n) = 0$ for all n, then

$$\begin{aligned}
h^{-1}(A) &= h^{-1}(\cup_{n\in\mathbb{N}} D(z_n, r_n)) \\
&\subset \cup_{n\in\mathbb{N}} h^{-1}(D(z_n, r_n)) \\
&= \cup_{n\in\mathbb{N}} E_n
\end{aligned}$$

which is a countable union of null sets, and therefore $h^{-1}(A)$ is a null set.

- next, we show that $h(x) \geq 1$ for $|\mu|$-a.e. x. Let $r \in (0, 1)$ and consider the inverse image of the closed ball

$$E^r = f^{-1}(D(0, r))$$

then we have for any partition $E^r = \sqcup_j E_j^r$

$$\begin{aligned}
\sum_j |\mu(E_j^r)| &= \sum_j \left| \int_{E_j^r} fd|\mu| \right| \\
&\leq \sum_j \int_{E_j^r} |f| d|\mu| \\
&\leq \sum_j \int_{E_j^r} rd|\mu| \\
&= r|\mu|(E^r)
\end{aligned}$$

therefore taking the sup over all such partitions, we have

$$|\mu|(E^r) \leq r|\mu|(E^r)$$

therefore $\mu(E^r) = 0$. We conclude by writing the open unit ball as the countable union of closed balls $B = \{z \in \mathbb{C} : |z| < 1\} = \cup_{n\in\mathbb{N}} D(0, 1 - 1/n)$, then

$$\begin{aligned}
h^{-1}(B) &= h^{-1}(\cup_{n\in\mathbb{N}} B(0, 1 - 1/n)) \\
&\subset \cup_{n\in\mathbb{N}} E^{(1-1/n)}
\end{aligned}$$

which is a countable union of null sets, therefore $h^{-1}(B)$ is a null set.

(8.2) Let X be a locally compact Hausdorff space. Regard $C_c(X)$ as a metric space, with metric

$$d(f,g) = \|f-g\|_{C_0} = \sup_{x\in X} |f(x)-g(x)|$$

Show that the completion of $C_c(X)$ is isometric to $C_0(X)$ (consider complex valued functions)

proof Let $(\overline{C_c(X)}, \bar{d})$ be the completion of $(C_c(X), d)$. An element of $\overline{C_c(X)}$ is an equivalence class for the equivalence relation

$$(f_n)\mathcal{R}(g_n) \Leftrightarrow \lim_n d(f_n, g_n) = 0$$

and the distance $\bar{d}$ is

$$\bar{d}([(f_n)],[(g_n)]) = \lim_n d(f_n, g_n)$$

Define the map

$$\begin{aligned} \phi : \overline{C_c(X)} &\to C_0(X) \\ [(f_n)] &\mapsto \phi([(f_n)]) = f \end{aligned}$$

here $[(f_n)]$ is an element of $\overline{C_c(X)}$, (f_n) is one Cauchy sequence in that equivalence class, and f is the limit of (f_n) in $C_0(X)$ (since $C_0(X)$ is complete).

- ϕ is well-defined since if (f_n) and (g_n) are both in $[(f_n)]$, then by definition $d(f_n, g_n) \to 0$. Therefore they have the same limit, since
$$d(g_n, f) \leq d(g_n, f_n) + d(f_n, f)$$
and both terms converge to 0.

- ϕ preserves distances: let $[(f_n)]$ and $[(g_n)]$ be two elements of $\overline{C_c(X)}$, and let $f = \phi([(f_n)])$ and $g = \phi([(g_n)])$. Then we have by the triangle inequality
$$d(f_n, g_n) \leq d(f, f_n) + d(f,g) + d(g, g_n)$$
and both $d(f, f_n)$ and $d(g, g_n)$ converge to 0, therefore taking the limit and using the definition of $\bar{d}$
$$\bar{d}([(f_n)],[(g_n)]) \leq d(f,g)$$
to show the reverse inequality, we use the triangle inequality again
$$d(f,g) \leq d(f, f_n) + d(f_n, g_n) + d(g, g_n)$$
where $(d(f_n, g_n))$ converges to $\bar{d}([(f_n)],[(g_n)])$, and $(d(f, f_n))$ and $(d(g, g_n))$ converge to 0. Taking the limit gives
$$d(f,g) \leq \bar{d}([(f_n)],[(g_n)])$$
hence equality of the distances. (a concise way to show it is to use continuity of d on $C_0(X)$). It follows in particular that ϕ is injective.

- ϕ is surjective: it suffices to show that $C_c(X)$ is dense in $C_0(X)$ (then for a given $f \in C_0(X)$, for all n, there exists $f_n \in C_c(X)$ such that $\|f_n - f\|_{C_0} \leq 1/n$. This defines a Cauchy sequence (f_n) of functions in $C_c(X)$, such that $\phi([(f_n)]) = f$). Let $f \in C_0(X)$ and $\epsilon > 0$. Since $f \in C_0(X)$. Then there exists a compact set K_1 such that $\forall x \notin K_1$, $f(x) < \epsilon$. Let
$$F_1 = f^{-1}([\epsilon, +\infty))$$
since f is continuous, F_1 is a closed. Then F_1 is a closed subset of the compact K_1, and is compact. Similarly,
$$F_2 = f^{-1}([2\epsilon, +\infty))$$

is compact, and we have

$$F_2 \subset \text{int}(F_1) \subset F_1$$

(to show this, we have $F_2 \subset O_{3/2} \subset F_1$ where $O_{3/2} = f^{-1}((\frac{3}{2}\epsilon, +\infty))$ is open). Now F_2 and $\text{int}(F_1)^c$ are closed, thus by Urysohn's lemma, there exists a continuous function $h : X \to [0,1]$ such that

$$h_{|F_2} \equiv 1$$
$$h_{|\,\text{int}(F_1)^c} \equiv 0$$

consider the function $g = fh \in C_c(X)$ (it is continuous as the product of two continuous functions, and has compact support since h has compact support). We have

- for all $x \in F_2$, $|g(x) - f(x)| = 0$
- for all $x \in X \setminus F_1$, $|g(x) - f(x)| = |-f(x)| \le \epsilon$ by definition of F_1
- for all $x \in F_1 \setminus F_2$, $|g(x) - f(x)| = |f(x)||1 - h(x)| \le 2\epsilon$ since $|f(x)| \le 2\epsilon$ on F_2^c and $|1 - h(x)| \le 1$.

therefore $\|g - f\|_{C_0} \le 2\epsilon$. This proves $C_c(X)$ is dense in $C_0(X)$, which completes the proof.

(8.3) Let μ and ν be σ-finite Radon measures on a locally compact Hausdorff space X. Consider the Lebesgue decomposition of ν with respect to μ

$$\nu = \rho + \tau$$

where ρ, τ are Borel measures, $\rho \perp \mu$, and $\tau \ll \mu$. Show that both ρ, τ are also Radon measures.

proof Since ρ and τ are positive measures, we have for all measurable E

$$\rho(E) \le \nu(E) \tag{2}$$

Since ν is σ-finite, there exists a partition $X = \sqcup_{n\in\mathbb{N}} X_n$ such that $\nu(X_n)$ is finite for all n.

- ρ is outer regular: let E be a measurable set, and let $\epsilon > 0$. Let $E_n = X_n \cap E$. Since $\nu(E_n)$ is finite, by outer regularity of ν there exists an open $O_n \supset E_n$ such that

  $$\nu(O_n \setminus E_n) \le 2^{-n}\epsilon$$

  Consider the open set $O = \cup_{n\in\mathbb{N}} O_n$. Then we have

  $$O \setminus E = (\cup_n O_n) \setminus (\cup_n E_n) \subset \cup_n O_n \setminus E_n$$

  therefore

  $$\begin{aligned} \rho(O \setminus E) &\le \nu(O \setminus E) && \text{by (2)} \\ &\le \sum_n \nu(O_n \setminus E_n) \\ &\le \sum_n 2^{-n}\epsilon \\ &\le \epsilon \end{aligned}$$

  which proves outer regularity.

- ρ is inner regular. Let E be a measurable set, and let $\epsilon > 0$. Let $E_n = X_n \cap E$. Since $\nu(E_n)$ is finite, by inner regularity of ν there exists an compact $K_n \subset E_n$ such that

  $$\nu(E_n \setminus K_n) \leq 2^{-n}\epsilon$$

  Consider the set $K = \cap_{n\in\mathbb{N}} K_n$. Then

  $$E \setminus K =\subset \cup_n E_n \setminus K_n$$

  and

  $$\begin{aligned} \rho(E \setminus K) &\leq \nu(E \setminus K) && \text{by (2)} \\ &\leq \sum_n \nu(E_n \setminus K_n) \\ &\leq \sum_n 2^{-n}\epsilon \\ &\leq \epsilon \end{aligned}$$

  thus

  $$\rho(K) \geq \rho(E) - \epsilon$$

  and $C_n = \cup_{i\leq n} K_i$ is a nested sequence of compact sets that converges to K, and by σ-additivity of ρ, we obtain

  $$\sup_{compact\ C\subset E} \rho(C) \geq \rho(E) - \epsilon$$

  since ϵ is arbitrary, this shows inner regularity.

(8.4) Let X be locally compact Hausdorff space. A sequence of functions is said to converge weakly to a limit $f \in C_0(X)$ if

$$\int_X f_n d\mu \to \int_X f d\mu$$

for every complex Radon measure μ on X.

1. Show that $f_n \to f$ weakly if $f_n(x) \to f(x)$ for every $x \in X$, and the sequence (f_n) is uniformly bounded.

   proof Let μ be a complex measure, and decompose $\mu = \mu_R + i\mu_I$. Then $|\mu_R(X)|$ and $\mu_I(X)$ are finite. Since (f_n) is uniformly bounded, there exists $M > 0$ such that for all n, $|f_n| \leq M$, and since $f \in C_0$, it is in particular bounded, and there exists M' such that $|f| \leq M'$. Thus $(f_n - f)$ is dominated by the integrable function $M + M'$. Therefore by the dominated convergence theorem,

   $$\begin{aligned} |\lim_n \int_X f_n - f d\mu_R| &\leq \lim_n \int_X |f_n - f| d\mu_R \\ &= \int_X \lim_n |f_n - f| d\mu_R && \text{by the DCT} \\ &= 0 && \text{by pointwise convergence} \end{aligned}$$

   similarly, $|\lim_n \int_X f_n - f d\mu_I| = 0$, and combining the two limits, we have the result.

2. Conversely, show that if $f_n, f \in C_0(X)$ and $f_n \to f$ weakly, then $f_n(x) \to f(x)$ for every $x \in X$, and (f_n) is uniformly bounded.

Pointwise convergence: let $x \in X$, and consider the measure δ_x defined by

$$\delta_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

δ_x is a Radon measure, since

- δ_x is a Borel measure
- inner regular: let E be a measurable set. Then if $x \in E$, then $\{x\}$ is a compact subset of E, and it follows that $\delta_x(E) = 1 = \sup_{compact K \subset E} \delta_x(K)$. If $x \notin E$, for any compact $K \subset E$, we have $x \notin K$, thus $\delta_x(K) = 0$ and $\sup_{compact K \subset E} \delta_x(K) = 0 = \delta_x(E)$, which proves inner regularity.
- outer regular: let E be a measurable set. Then if $x \in E$, then for any open $O \supset E$, $x \in O$, thus $\delta_x(E) = \inf_{open O \supset E} \delta_x(O)$. If $x \notin E$, then there exists an open set $O \supset E$ such that $x \notin O$ (since the space is Hausdorff, for all $y \in E$, there exists open O_y that contains y but not x. Then $O = \cup_{y \in E} O_y$ is an open superset of E, and does not contain x), and it follows that $\inf_{open O \supset E} \delta_x(O) = 0 = \delta_x(E)$.

We also have for any measurable function h,

$$\begin{aligned} \int_X h d\delta_x &= \int_{\{x\}} h d\delta_x && \text{since } X \setminus \{x\} \text{ has measure } 0 \\ &= h(x) \end{aligned}$$

$\delta_x(X)$ is also finite, therefore δ_x is a complex measure, thus $\int_X f_n d\delta_x \to \int_X f d\delta_x$, i.e.

$$\lim_n f_n(x) = f(x)$$

(f_n) is uniformly bounded: by contradiction, assume not. Then for all $A > 0$, there exist infinitely many $n \in \mathbb{N}$ and $x_n \in X$ such that $|f_n(x_n)| > A$. Since f is C_0, it is in particular bounded. Let $M > 0$ be a bound on $|f|$.

Construct a subsequence (f_{n_k}), and a sequence of points (x_k) as follows:

- let $n_1 = 1$, and x_1 such that $|f_{n_1}(x_1)| > 1$ for all k, choose $n_k > n_{k-1}$, and $x_k \in X$ such that

$$\begin{aligned} |f_{n_k}(x_i)| &\le M + 1 && \forall i < k \\ |f_{n_k}(x_k)| &> 4^k(M_{k-1} + 1) \end{aligned}$$

  where

$$M_{k-1} = \max\{\|f_{n_1}\|_{C_0}, \dots, \|f_{n_{k-1}}\|_{C_0}, M\}$$

  (since for all $i < k$, $(f_n(x_i))_n$ converges to $f(x_i)$, there exists N_i such that for all $n > N_i$, $\|f_n(x_i) - f(x_I)\| \le 1$, thus $\|f_n(x_i)\| \le |f| + 1 \le M + 1$, then by non-uniform boundedness, there exists $n \ge \max\{N_1, \dots, N_{k-1}, n_{k-1}\}$ and x such that $|f_n(x_k)| > 4^k M_{k-1}$).
  Now define the measure

$$\mu = \sum_{k=1}^{\infty} c_k \delta_{x_k}$$

  where $c_1 = 1$ and $c_k = \frac{2^{-k}}{1+M_{k-1}} \le 2^{-k}$. μ is a Radon measure. Let E be a measurable set. Then $\mu(E) = \sum_{k: x_k \in E} c_k$. Let $\epsilon > 0$, and let N_ϵ such that $\sum_{k \ge N_\epsilon} c_k \le \epsilon$

- inner regularity: consider $K = \{x_1, \dots x_{N_\epsilon - 1}\} \cap E$ is compact (a finite set), and $\mu(E) - \mu(K) \leq \sum_{k \geq N_\epsilon} c_k \leq \epsilon$.

We also have $\mu(X) = \sum_{k=1}^\infty c_k \leq \sum_{k=1}^\infty 2^{-k} \leq 1$ (thus it is also a complex measure). By weak convergence, we have

$$\lim_n \int_X (f_n - f) d\mu = 0$$

thus the subsequence $\left(\int_X (f_{n_i} - f) d\mu\right)_i$ also converges to 0. Yet, we have

$$\begin{aligned} |\int_X (f_{n_i} - f) d\mu| &= |\sum_{k=1}^\infty c_k (f_{n_i}(x_k) - f(x_k))| \\ &\geq |c_k||f_{n_k}(x_k)| - \sum_{k=1}^{i-1} |c_k||f_{n_i}(x_k)| - \sum_{k=i+1}^\infty |c_i||f_{n_i}(x_k)| - \sum_{k=1}^\infty |c_k||f(x_k)| \quad \text{by the triangle inequalit} \end{aligned}$$

now using the properties of (f_{n_k}) and x_k, we have for all $k > 1$

$$\begin{aligned} |c_k||f_{n_k}(x_k)| &> \frac{2^{-k}}{M_{k-1}+1} 4^k (M_{k-1}+1) = 2^k && \text{for } i = k \\ |c_k||f_{n_i}(x_k)| &\leq \frac{2^{-k}}{M_{k-1}+1} (M+1) \leq 2^{-k} && \text{for } k < i \text{ by construction of } f_{n_i} \\ |c_k||f_{n_i}(x_k)| &\leq \frac{2^{-k}}{M_{k-1}+1} |f_{n_i}| \leq 2^{-k} && \text{for } k > i \text{ by definition of } c_k \\ |f| &\leq M \end{aligned}$$

combining these inequalities, we have for all $k > 1$

$$\begin{aligned} |\int_X (f_{n_i} - f) d\mu| &\geq 2^k - \sum_{k=1}^{i-1} 2^{-k} - \sum_{k=i+1}^\infty 2^{-k} - \sum_{k=1}^\infty 2^{-k} \\ &\geq 2^k - 2 \end{aligned}$$

which contradicts convergence of the subsequence $(\int_X (f_{n_i} - f) d\mu)_i$ to 0.

(8.5) Let X be a compact Hausdorff space. In this problem we show that there is a meaningful notion of the support of a Radon measure on X.

Let μ be a Radon measure on X. Show that there exists a compact set $K \subset X$ such that $\mu(K) = \mu(X)$, but $\mu(K') < \mu(K)$ for every proper compact subset $K' \subset K$.

proof Let $\mathcal{O}$ be the collection of open null subsets of X, and consider

$$O = \cup_{U \in \mathcal{O}} U$$

then O is open, and O is a null set. Indeed, for all compact $F \subset O$, $\mathcal{O}$ is an open cover of F since $\cup_{U\in\mathcal{O}} U = O \supset F$. Thus there exists a finite subcover, i.e. there exist $U_1, \ldots, U_n \in \mathcal{O}$ such that

$$\cup_{i=1}^n U_i \supset F$$

thus we have

$$\mu(F) = \mu(\cup_{i=1}^n U_i) \le \sum_{i=1}^n \mu(U_i) = 0$$

since every $U_i \in \mathcal{O}$ is a null set. Therefore by inner regularity of μ on open sets,

$$\mu(O) = \sup_{\text{compact } F\subset O} \mu(F) = 0$$

Now let $K = X\setminus O$. K is compact since it is closed subset of the compact X, and $\mu(K) = \mu(X\setminus O) = \mu(X$ since O is a null set.

Now let K' be a proper subset of K. Consider the open set $X \setminus K'$. $X \setminus K'$ is not a null set, for otherwise we would have $X \setminus K' \subset O$ (O is the union of all open null sets), i.e. $K' \supset X \setminus O = K$, and this would contradict the fact that K' is a proper subset of K. Therefore

$$\begin{aligned} \mu(X) - \mu(K') &= \mu(X \setminus K) && \text{since } \mu(X) \text{ is finite} \\ &> 0 && \text{since } X \setminus K \text{ is not a null set} \end{aligned}$$

which proves $\mu(K') < \mu(X)$.