Let $\mu$ be a complex measure on $(X, A)$. The associated positive measure $|\mu|$ is defined by, for all $E \in A$

$$|\mu|(E) = \sup \sum |\mu(E_j)|$$

where the sup is taken over all decompositions of $E$ as a union of countably many pairwise disjoint measurable subsets $E_j$. Show that there exists a measurable function $h$ satisfying $|h| = 1$ $|\mu|$-almost everywhere, such that $\mu = h|\mu|$, that is for all $E \in A$,

$$\mu(E) = \int_E h d|\mu|$$

**proof** In order to apply the Radon-Nykovym theorem, we first show that $|\mu|(X)$ is finite. Decompose the complex measure $\mu$ into

$$\mu = (\mu_R^+ - \mu_R^-) + i(\mu_I^+ - \mu_I^-)$$

where all measures in the decomposition are (positive) finite Radon measures. Then we have for all measurable $E$, using the fact that for any complex $z = a + ib$, $|z| \leq |a| + |b|

$$|\mu(E)| \leq |\mu_R(E)| + |\mu_I(E)| = \mu_R^+(E) + \mu_R^-(E) + \mu_I^+(E) + \mu_I^-(E)$$

therefore for any partition $X = \bigsqcup_i E_i$,

$$\sum_i |\mu(E_i)| \leq \sum_i (\mu_R^+(E_i) + \mu_R^-(E_i) + \mu_I^+(E_i) + \mu_I^-(E_i))$$

$$= \mu_R^+(X) + \mu_R^-(X) + \mu_I^+(X) + \mu_I^-(X)$$

which is finite. Taking the sup over all such partitions, we obtain $|\mu|(X)$ is finite.

Next, we have by definition of $|\mu|$, for all measurable $E$,

$$|\mu|(E) \geq |\mu(E)|$$

(1)

since the sup in the definition of $|\mu|$ is taken over all partitions, in particular the partition that contains a single element $\{E\}$.

We have in particular $\mu << |\mu|$, and by the Radon-Nikodym Theorem, there exists a measurable function $h$ such that for all measurable $E$

$$\mu(E) = \int_E h d|\mu|$$

• first, we show that $|h(x)| \leq 1$ for $|\mu|$-a.e. $x \in X$. Let $A = \{z \in \mathbb{C} : |z| > 1\}$. We seek to show that $h^{-1}(A)$ is a null set. Since $A$ is an open subset of $\mathbb{C}$, it is the union of countably many closed balls

$$A = \cup_{n \in \mathbb{N}} D(z_n, r_n)$$
let $E_n = h^{-1}(D(z_n, r_n))$. $E_n$ is measurable, and a null set, for otherwise we would have

$$\left| \frac{\int_{E_n} hd\mu}{\mu(E_n)} - z_n \right| = \left| \frac{\int_{E_n} h d\mu}{\mu(E_n)} - z_n \right|$$

$$= \frac{1}{\mu(E_n)} \left| \int_{E_n} (h - z_n) d\mu \right|$$

$$\leq \frac{1}{\mu(E_n)} \int_{E_n} |h - z_n| d\mu$$

$$\leq \frac{1}{\mu(E_n)} \int_{E_n} r_n d\mu$$

since $h(E_n) = D(z_n, r_n)$

$$= r_n$$

and it would follow that $\frac{\mu(E_n)}{\mu(E_n)} \in D(z_n, r_n) \subset A$ which contradicts the fact $|\mu(E_n)| \leq |\mu|(E_n)$ in (1).

Therefore $|\mu|(E_n) = 0$ for all $n$, then

$$h^{-1}(A) = h^{-1}(\bigcup_{n \in \mathbb{N}} D(z_n, r_n))$$

$$\subset \bigcup_{n \in \mathbb{N}} h^{-1}(D(z_n, r_n))$$

$$= \bigcup_{n \in \mathbb{N}} E_n$$

which is a countable union of null sets, and therefore $h^{-1}(A)$ is a null set.

• next, we show that $h(x) \geq 1$ for $|\mu|$-a.e. $x$. Let $r \in (0, 1)$ and consider the inverse image of the closed ball

$$E^r = f^{-1}(D(0, r))$$

then we have for any partition $E^r = \bigcup_j E_j^r$

$$\sum_j |\mu(E_j^r)| = \sum_j \left| \int_{E_j^r} f d\mu \right|$$

$$\leq \sum_j \int_{E_j^r} |f| d\mu$$

$$\leq \sum_j \int_{E_j^r} r d\mu$$

$$= r|\mu|(E^r)$$

therefore taking the sup over all such partitions, we have

$$|\mu|(E^r) \leq r|\mu|(E^r)$$

therefore $\mu(E^r) = 0$. We conclude by writing the open unit ball as the countable union of closed balls

$$B = \{ z \in \mathbb{C} : |z| < 1 \} = \bigcup_{n \in \mathbb{N}} D(0, 1 - 1/n)$$

then

$$h^{-1}(B) = h^{-1}(\bigcup_{n \in \mathbb{N}} D(0, 1 - 1/n))$$

$$\subset \bigcup_{n \in \mathbb{N}} h^{-1}(D(1 - 1/n))$$

which is a countable union of null sets, therefore $h^{-1}(B)$ is a null set.
(8.2) Let $X$ be a locally compact Hausdorff space. Regard $C_c(X)$ as a metric space, with metric
\[ d(f, g) = \|f - g\|_{C_0} = \sup_{x \in X} |f(x) - g(x)| \]
Show that the completion of $C_c(X)$ is isometric to $C_0(X)$ (consider complex valued functions)

**proof** Let $(\overline{C_c(X)}, \overline{d})$ be the completion of $(C_c(X), d)$. An element of $\overline{C_c(X)}$ is an equivalence class for the equivalence relation
\[ (f_n) \sim (g_n) \iff \lim_{n} d(f_n, g_n) = 0 \]
and the distance $\overline{d}$ is
\[ \overline{d}([[f_n]], [[g_n]]) = \lim_{n} d(f_n, g_n) \]
Define the map
\[ \phi : \overline{C_c(X)} \to C_0(X) \]
\[ [[f_n]] \mapsto \phi([[f_n]]) = f \]
here $[[f_n]]$ is an element of $\overline{C_c(X)}$, $(f_n)$ is one Cauchy sequence in that equivalence class, and $f$ is the limit of $(f_n)$ in $C_0(X)$ (since $C_0(X)$ is complete).

- $\phi$ is well-defined since if $(f_n)$ and $(g_n)$ are both in $[[f_n]]$, then by definition $d(f_n, g_n) \to 0$. Therefore they have the same limit, since
\[ d(g_n, f) \leq d(g_n, f_n) + d(f_n, f) \]
and both terms converge to 0.

- $\phi$ preserves distances: let $[[f_n]]$ and $[[g_n]]$ be two elements of $\overline{C_c(X)}$, and let $f = \phi([[f_n]])$ and $g = \phi([[g_n]])$. Then we have by the triangle inequality
\[ d(f_n, g_n) \leq d(f, f_n) + d(f, g) + d(g, g_n) \]
and both $d(f, f_n)$ and $d(g, g_n)$ converge to 0, therefore taking the limit and using the definition of $\overline{d}$
\[ \overline{d}([[f_n]], [[g_n]]) \leq d(f, g) \]
to show the reverse inequality, we use the triangle inequality again
\[ d(f, g) \leq d(f, f_n) + d(f_n, g_n) + d(g, g_n) \]
where $(d(f_n, g_n))$ converges to $\overline{d}([[f_n]], [[g_n]])$, and $(d(f, f_n))$ and $(d(g, g_n))$ converge to 0. Taking the limit gives
\[ d(f, g) \leq \overline{d}([[f_n]], [[g_n]]) \]
hence equality of the distances. (a concise way to show it is to use continuity of $d$ on $C_0(X)$). It follows in particular that $\phi$ is injective.

- $\phi$ is surjective: it suffices to show that $C_c(X)$ is dense in $C_0(X)$ (then for a given $f \in C_0(X)$, for all $n$, there exists $f_n \in C_c(X)$ such that $\|f_n - f\|_{C_0} \leq 1/n$. This defines a Cauchy sequence $(f_n)$ of functions in $C_c(X)$, such that $\phi([[f_n]]) = f$). Let $f \in C_0(X)$ and $\epsilon > 0$. Since $f \in C_0(X)$. Then there exists a compact set $K_1$ such that $\forall x \notin K_1$, $f(x) < \epsilon$. Let
\[ F_1 = f^{-1}([\epsilon, +\infty)) \]
since $f$ is continuous, $F_1$ is a closed. Then $F_1$ is a closed subset of the compact $K_1$, and is compact. Similarly,
\[ F_2 = f^{-1}([2\epsilon, +\infty)) \]
is compact, and we have
\[ F_2 \subset \text{int}(F_1) \subset F_1 \]
(to show this, we have \( F_2 \subset O_{3/2} \subset F_1 \) where \( O_{3/2} = f^{-1}\left((\frac{3}{2}\epsilon, +\infty)\right) \) is open). Now \( F_2 \) and \( \text{int}(F_1) \) are closed, thus by Urysohn’s lemma, there exists a continuous function \( h : X \to [0, 1] \) such that
\[
\begin{align*}
h|_{F_2} & \equiv 1 \\
h|_{\text{int}(F_1)^c} & \equiv 0
\end{align*}
\]
consider the function \( g = fh \in C_c(X) \) (it is continuous as the product of two continuous functions, and has compact support since \( h \) has compact support). We have

- for all \( x \in F_2 \), \(|g(x) - f(x)| = 0\)
- for all \( x \in X \setminus F_1 \), \(|g(x) - f(x)| = |f(x)| \leq \epsilon \) by definition of \( F_1 \)
- for all \( x \in F_1 \setminus F_2 \), \(|g(x) - f(x)| = |f(x)| |1 - h(x)| \leq 2\epsilon \) since \(|f(x)| \leq 2\epsilon \) on \( F_2^c \) and \(|1 - h(x)| \leq 1\).

therefore \( \|g - f\|_{C_0} \leq 2\epsilon \). This proves \( C_c(X) \) is dense in \( C_0(X) \), which completes the proof.

\[ (8.3) \] Let \( \mu \) and \( \nu \) be \( \sigma \)-finite Radon measures on a locally compact Hausdorff space \( X \). Consider the Lebesgue decomposition of \( \nu \) with respect to \( \mu \)
\[ \nu = \rho + \tau \]
where \( \rho, \tau \) are Borel measures, \( \rho \perp \mu \), and \( \tau \ll \mu \). Show that both \( \rho, \tau \) are also Radon measures.

**proof** Since \( \rho \) and \( \tau \) are positive measures, we have for all measurable \( E \)
\[ \rho(E) \leq \nu(E) \] (2)
Since \( \nu \) is \( \sigma \)-finite, there exists a partition \( X = \bigcup_{n \in \mathbb{N}} X_n \) such that \( \nu(X_n) \) is finite for all \( n \).

- \( \rho \) is outer regular: let \( E \) be a measurable set, and let \( \epsilon > 0 \). Let \( E_n = X_n \cap E \). Since \( \nu(E_n) \) is finite, by outer regularity of \( \nu \) there exists an open \( O_n \supset E_n \) such that
\[ \nu(O_n \setminus E_n) \leq 2^{-n} \epsilon \]
Consider the open set \( O = \bigcup_{n \in \mathbb{N}} O_n \). Then we have
\[ O \setminus E = (\bigcup_{n \in \mathbb{N}} O_n) \setminus (\bigcup_{n \in \mathbb{N}} E_n) \subset \bigcup_{n \in \mathbb{N}} O_n \setminus E_n \]
therefore
\[
\begin{align*}
\rho(O \setminus E) & \leq \nu(O \setminus E) \\
& \leq \sum_n \nu(O_n \setminus E_n) \\
& \leq \sum_n 2^{-n} \epsilon \\
& \leq \epsilon
\end{align*}
\]
which proves outer regularity.
• \( \rho \) is inner regular. Let \( E \) be a measurable set, and let \( \epsilon > 0 \). Let \( E_n = X_n \cap E \). Since \( \nu(E_n) \) is finite, by inner regularity of \( \nu \) there exists an compact \( K_n \subset E_n \) such that

\[ \nu(E_n \setminus K_n) \leq 2^{-n} \epsilon \]

Consider the set \( K = \cap_{n \in \mathbb{N}} K_n \). Then

\[ E \setminus K \subset \cup_n E_n \setminus K_n \]

and

\[ \rho(E \setminus K) \leq \nu(E \setminus K) \leq \sum_n \nu(E_n \setminus K_n) \leq \sum_n 2^{-n} \epsilon \leq \epsilon \]

thus

\[ \rho(K) \geq \rho(E) - \epsilon \]

and \( C_n = \cup_{i \leq n} K_i \) is a nested sequence of compact sets that converges to \( K \), and by \( \sigma \)-additivity of \( \rho \), we obtain

\[ \sup_{\text{compact } C \subset E} \rho(C) \geq \rho(E) - \epsilon \]

since \( \epsilon \) is arbitrary, this shows inner regularity.

(8.4) Let \( X \) be locally compact Hausdorff space. A sequence of functions is said to converge weakly to a limit \( f \in C_0(X) \) if

\[ \int_X f_n d\mu \to \int_X f d\mu \]

for every complex Radon measure \( \mu \) on \( X \).

1. Show that \( f_n \to f \) weakly if \( f_n(x) \to f(x) \) for every \( x \in X \), and the sequence \( (f_n) \) is uniformly bounded.

**proof** Let \( \mu \) be a complex measure, and decompose \( \mu = \mu_R + i\mu_I \). Then \( |\mu_R(X)| \) and \( \mu_I(X) \) are finite. Since \( (f_n) \) is uniformly bounded, there exists \( M > 0 \) such that for all \( n \), \( |f_n| \leq M \), and since \( f \in C_0 \), it is in particular bounded, and there exists \( M' \) such that \( |f| \leq M' \). Thus \( (f_n - f) \) is dominated by the integrable function \( M + M' \). Therefore by the dominated convergence theorem,

\[ |\lim_n \int_X f_n - f d\mu_R| \leq \lim_n \int_X |f_n - f| d\mu_R \]

\[ = \int_X \lim_n |f_n - f| d\mu_R \]

\[ = 0 \]

by the DCT

similarly, \( |\lim_n \int_X f_n - f d\mu_I| = 0 \), and combining the two limits, we have the result.
2. Conversely, show that if \( f_n, f \in C_0(X) \) and \( f_n \to f \) weakly, then \( f_n(x) \to f(x) \) for every \( x \in X \), and \( (f_n) \) is uniformly bounded.

Pointwise convergence: let \( x \in X \), and consider the measure \( \delta_x \) defined by

\[
\delta_x(E) = \begin{cases} 
1 & \text{if } x \in E \\
0 & \text{otherwise}
\end{cases}
\]

\( \delta_x \) is a Radon measure, since

- **\( \delta_x \) is a Borel measure**
- **inner regular**: let \( E \) be a measurable set. Then if \( x \in E \), then \( \{x\} \) is a compact subset of \( E \), and it follows that \( \delta_x(E) = 1 = \sup_{\text{compact } K \subset E} \delta_x(K) \). If \( x \notin E \), for any compact \( K \subset E \), we have \( x \notin K \), thus \( \delta_x(K) = 0 \) and \( \sup_{\text{compact } K \subset E} \delta_x(K) = 0 = \delta_x(E) \), which proves inner regularity.
- **outer regular**: let \( E \) be a measurable set. Then if \( x \in E \), then for any open \( O \supset E, x \in O \), thus \( \delta_x(E) = \inf_{\text{open } O \supset E} \delta_x(O) \). If \( x \notin E \), then there exists an open set \( O \supset E \) such that \( x \notin O \) (since the space is Hausdorff, for all \( y \in E \), there exists open \( O_y \) that contains \( y \) but not \( x \). Then \( O = \bigcup_{y \in E} O_y \) is an open superset of \( E \), and does not contain \( x \), and it follows that \( \inf_{\text{open } O \supset E} \delta_x(O) = 0 = \delta_x(E) \).

We also have for any measurable function \( h \),

\[
\int_X h \, d\delta_x = \int_{\{x\}} h \, d\delta_x = h(x) \quad \text{since } X \setminus \{x\} \text{ has measure 0}
\]

\( \delta_x(X) \) is also finite, therefore \( \delta_x \) is a complex measure, thus \( \int_X f_n \, d\delta_x \to \int_X f \, d\delta_x \), i.e.

\[
\lim_n f_n(x) = f(x)
\]

\( (f_n) \) is uniformly bounded: by contradiction, assume not. Then for all \( A > 0 \), there exist infinitely many \( n \in \mathbb{N} \) and \( x_n \in X \) such that \( |f_n(x_n)| > A \). Since \( f \) is \( C_0 \), it is in particular bounded. Let \( M > 0 \) be a bound on \( |f| \).

Construct a subsequence \( (f_{n_k}) \), and a sequence of points \( (x_k) \) as follows:

- let \( n_1 = 1 \), and \( x_1 \) such that \( |f_n(x_1)| > 1 \) for all \( k \), choose \( n_k > n_{k-1} \), and \( x_k \in X \) such that

\[
\begin{align*}
|f_{n_k}(x_i)| &\leq M + 1 & \forall i < k \\
|f_{n_k}(x_k)| &> 4^k(M_{k-1} + 1)
\end{align*}
\]

where

\[
M_{k-1} = \max\{\|f_{n_1}\|_{C_0}, \ldots, \|f_{n_{k-1}}\|_{C_0}, M\}
\]

(since for all \( i < k \), \( (f_n(x_i))_n \) converges to \( f(x_i) \), there exists \( N_i \) such that for all \( n > N_i \), \( \|f_n(x_i) - f(x_i)\| \leq 1 \), thus \( \|f_n(x_i)\| \leq |f| + 1 \leq M + 1 \), then by non-uniform boundedness, there exists \( n \geq \max\{N_1, \ldots, N_{k-1}, n_{k-1}\} \) and \( x \) such that \( |f_n(x_k)| > 4^kM_{k-1} \).

Now define the measure

\[
\mu = \sum_{k=1}^{\infty} c_k \delta_{x_k}
\]

where \( c_1 = 1 \) and \( c_k = \frac{2^{-k}}{1 + M_{k-1}} \leq 2^{-k} \). \( \mu \) is a Radon measure. Let \( E \) be a measurable set. Then

\[
\mu(E) = \sum_{k: x_k \in E} c_k. \quad \text{Let } \epsilon > 0 \text{, and let } N_\epsilon \text{ such that } \sum_{k \geq N_\epsilon} c_k \leq \epsilon
\]

6
inner regularity: consider $K = \{x_1, \ldots, x_{N-1}\} \cap E$ is compact (a finite set), and $\mu(E) - \mu(K) \leq \sum_{k \geq N} c_k \leq \epsilon$.

We also have $\mu(X) = \sum_{k=1}^{\infty} c_k \leq \sum_{k=1}^{\infty} 2^{-k} \leq 1$ (thus it is also a complex measure). By weak convergence, we have

$$\lim_n \int_X (f_n - f) d\mu = 0$$

thus the subsequence $(\int_X (f_{n_i} - f) d\mu)_i$ also converges to 0. Yet, we have

$$| \int_X (f_{n_i} - f) d\mu | \geq \sum_{k=1}^{i-1} |c_k||f_{n_i}(x_k)| - \sum_{k=i+1}^{\infty} |c_k||f_{n_i}(x_k)| \geq 2^{k-1}$$

by the triangle inequality.

now using the properties of $(f_{n_k})$ and $x_k$, we have for all $k > 1$

$$|c_k||f_{n_k}(x_k)| > \frac{2^{-k}}{M_{k-1} + 1} 4^k (M_{k-1} + 1) = 2^k$$

for $i = k$

$$|c_k||f_{n_i}(x_k)| \leq \frac{2^{-k}}{M_{k-1} + 1} (M + 1) \leq 2^{-k}$$

for $k < i$ by construction of $f_{n_i}$

$$|c_k||f_{n_i}(x_k)| \leq \frac{2^{-k}}{M_{k-1} + 1} |f_{n_i}| \leq 2^{-k}$$

for $k > i$ by definition of $c_k$

$$|f| \leq M$$

combining these inequalities, we have for all $k > 1$

$$| \int_X (f_{n_i} - f) d\mu | \geq 2^k - \sum_{k=1}^{i-1} 2^{-k} - \sum_{k=i+1}^{\infty} 2^{-k} - \sum_{k=1}^{\infty} 2^{-k}$$

which contradicts convergence of the subsequence $(\int_X (f_{n_i} - f) d\mu)_i$ to 0.

(8.5) Let $X$ be a compact Hausdorff space. In this problem we show that there is a meaningful notion of the support of a Radon measure on $X$.

Let $\mu$ be a Radon measure on $X$. Show that there exists a compact set $K \subset X$ such that $\mu(K) = \mu(X)$, but $\mu(K') < \mu(K)$ for every proper compact subset $K' \subset K$.

**proof** Let $O$ be the collection of open null subsets of $X$, and consider

$$O = \cup_{U \in O} U$$

---

7
then $O$ is open, and $O$ is a null set. Indeed, for all compact $F \subset O$, $O$ is an open cover of $F$ since $\bigcup_{U \in O} U = O \supset F$. Thus there exists a finite subcover, i.e. there exist $U_1, \ldots, U_n \in O$ such that

$$\bigcup_{i=1}^n U_i \supset F$$

thus we have

$$\mu(F) = \mu\left(\bigcup_{i=1}^n U_i\right) \leq \sum_{i=1}^n \mu(U_i) = 0$$

since every $U_i \in O$ is a null set. Therefore by inner regularity of $\mu$ on open sets,

$$\mu(O) = \sup_{\text{compact } F \subset O} \mu(F) = 0$$

Now let $K = X \setminus O$. $K$ is compact since it is closed subset of the compact $X$, and $\mu(K) = \mu(X \setminus O) = \mu(X)$ since $O$ is a null set.

Now let $K'$ be a proper subset of $K$. Consider the open set $X \setminus K'$. $X \setminus K'$ is not a null set, for otherwise we would have $X \setminus K' \subset O$ ($O$ is the union of all open null sets), i.e. $K' \supset X \setminus O = K$, and this would contradict the fact that $K'$ is a proper subset of $K$. Therefore

$$\mu(X) - \mu(K') = \mu(X \setminus K)$$

since $\mu(X)$ is finite

$$> 0$$

since $X \setminus K$ is not a null set

which proves $\mu(K') < \mu(X)$. 