MATH 202B - Problem Set 6

Walid Krichene (23265217)

March 22, 2013

(6.1) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces. Let $f : X \times Y \to \mathbb{R}^*$ be a measurable function. Let $f_x(y) = f(x, y)$. For all $x \in X$, define

$$F(x) = \|f^x\|_{L^\infty(Y)}$$

Show that F is measurable with respect to $\mathcal{A}$.

proof Consider the case where f is the indicator function of a measurable set E, $f = 1_E$. Then we have

$$F(x) = \|1_{E^x}\|_{L^\infty(Y)} = \begin{cases} 1 & \text{if } \nu(E^x) > 0 \\ 0 & \text{if } \nu(E^x) = 0 \end{cases}$$

and since $g : x \mapsto \nu(E^x)$ is μ-measurable, it follows that F is measurable since

$$F^{-1}(-\infty, c) = \begin{cases} \emptyset & \text{if } c < 0 \\ g^{-1}(\{0\}) & \text{if } 0 \le c < 1 \\ X & \text{if } 1 \le c \end{cases}$$

Now if f is a simple function of the form $f = \sum_{n=1}^N c_n 1_{E_n}$ where E_n are measurable and disjoint, then we have

$$\begin{aligned} F(x) &= \|\sum_n c_n 1_{E_n^x}\|_{L^\infty(Y)} \\ &= \max_n |c_n| \|1_{E_n^x}\|_{L^\infty(Y)} && \text{since the } E_n \text{ are disjoint} \\ &= \max_n |c_n| F_n(x) \end{aligned}$$

where $F_n(x)$ is the function associated to the simple function 1_{E_n}, and was shown to be measurable. F is thus measurable as the maximum of finitely many measurable functions.

Now consider the general case where f is a general measurable function, and let f_n be a sequence of simple functions that converge to f pointwise, such that for all n, $f_n(x, y) \le f_{n+1}(x, y) \le f(x, y)$. For all n, associate F_n to f_n. Then we have for all x, $f_n^x \le f_{n+1}^x \le f^x$, thus

$$F_n(x) \le F_{n+1}(x) \le F(x)$$

thus $(F_n(x))$ is a monotone bounded sequence, thus it converges, and

$$\lim_n F_n(x) \le F(x)$$

to prove equality, let $\epsilon > 0$. Then $\{y : |f^x(y)| \ge F(x) - \epsilon\}$ has positive measure.

...

Therefore $F(x) = \lim_n F_n(x)$, and F is the limit of measurable functions, thus it is measurable.

(6.2) Let $(X, \mathcal{A}, \mu)$ be a $\sigma-$finite measure space. Let $\mathcal{B} \subset \mathcal{A}$ be a $\sigma-$algebra, and suppose $(X, \mathcal{B}, \mu_{|\mathcal{B}})$ is also σ-finite.
(i) Show that for any $f \in L^1(X, \mathcal{A}, \mu)$ there exists a function $g \in L^1(X, \mathcal{B}, \mu_{|\mathcal{B}})$ such that for all $E \in \mathcal{B}$, $\int_E g d\mu = \int_E f d\mu$. This function g is denoted $\mathrm{E}[f|\mathcal{B}]$
(ii) Work out what this construction means in the following special case: $(\mathbb{R}, \mathcal{A}, m)$ where $\mathcal{A}$ is the set of Lebesgue measurable sets, and $\mathcal{B}$ is the σ-algebra generated by all intervals $[n, n+1)$, $n \in \mathbb{Z}$.

proof (i) Define

$$\rho : \mathcal{A} \to \mathbb{R}$$
$$A \mapsto \int_A f d\mu$$

since f is $\mathcal{A}$-measurable, ρ is a finite measure on $\mathcal{A}$. Consider the measure $\rho_{|\mathcal{B}}$. $\rho_{|\mathcal{B}} \ll \mu_{|\mathcal{B}}$ since for all μ-null sets $B \in \mathcal{B}$, $\rho_{|\mathcal{B}}(B = \int_B f d\mu = 0$. By the Radon-Nykodym theorem, there exists a $\mathcal{B}-$ measurable function g, $g \in L^1(X, \mathcal{B}, \mu_{|\mathcal{B}})$ such that for all $E \in \mathcal{B}$,

$$\rho(E) = \int_E g d\mu$$

and by definition of ρ, $\rho(E) = \int_E f d\mu$. This proves that g satisfies the desired property.
(ii) The $\mathcal{B}$-measurable function g is in this case given by a sequence (a_n), such that $\sum_{n\in\mathbb{Z}} |a_n| < \infty$, and this sequence satisfies: for all subsets $N \subset \mathbb{Z}$,

$$\int_{\cup_{n\in N}[n,n+1)} f = \sum_{n\in N} a_n$$

(6.3) Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ be σ algebras. Let $f \in L^1(X, \mathcal{A}, \mu)$. Prove the following

1. $\mathrm{E}[\mathrm{E}[f|\mathcal{B}]|\mathcal{C}] = \mathrm{E}[f|\mathcal{C}]$ almost everywhere.

   proof Let $g = \mathrm{E}[f|\mathcal{B}]$, $h = \mathrm{E}[g|\mathcal{C}]$ and $i = \mathrm{E}[f|\mathcal{C}]$. Then we have for all $E \in \mathcal{C}$, $\int_E f d\mu = \int_E i d\mu$ and $\int_E h d\mu = \int_E g d\mu$. And since E is also in $\mathcal{B}$, we also have $\int_E g d\mu = \int_E h f \mu$. Therefore for all $E \in \mathcal{C}$,

   $$\int_E h d\mu = \int_E i d\mu$$

   therefore $h = i$ $\mu_{|\mathcal{C}}$-almost everywhere, by the following

   Lemma 1 *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. If f, g are two measurable functions such that for all $E \in \mathcal{A}$, $\int_E f d\mu = \int_E g d\mu$, then $f = g$ a.e.*

   indeed, first prove that $E = \{x : f(x) > g(x)\}$ is a null set, by writing $E = \cup_{k \in \mathbb{N}} E_k$ where $E_k = \{x : f(x) - g(x) > 1/k\}$. Then $0 = \int_{E_k}(f - g) \geq \frac{1}{k}\mu(E_k)$. Therefore $\mu(E_k) = 0$ for all k, and $\mu(\cup_k E_k) = 0$. By symmetry, we also have $\{x : f(x) < g(x)\}$ is a null set, which completes the proof of the lemma.

2. Let $p \in [1, \infty]$ If $f \in L^1 \cap L^p$, then $\mathrm{E}[f|\mathcal{B}] \in (L^1 \cap L^p)(X, \mathcal{B}, \mu_{|\mathcal{B}})$

   proof Consider the case $p < \infty$, and assume the space has finite measure (we will generalize the result later).

   Given a convex function $\phi : \mathbb{R} \to \mathbb{R}$, we show that $\phi(\mathrm{E}[f|\mathcal{B}](x)) \leq \mathrm{E}[\phi(f)](x)$ for a.e. x. To prove this fact (Jensen's inequality for the expectation), consider the set of affine functions with rational coefficients that under approximate ϕ:

   $$A = \{(a, b) \in \mathbb{Q} \times \mathbb{Q} : au + b \leq \phi(u) \forall u\}$$

   then we have

   $$\forall u \in \mathbb{R},\ \phi(u) = \sup_{(a,b) \in A} (au + b)$$

   We have for all $(a, b) \in A$ and for all x

   $$\phi(f(x)) \geq af(x) + b$$

   taking the expectation and by linearity of the expectation (and the fact that if $f \leq g$ then $\mathrm{E}[f|\mathcal{B}] \leq \mathrm{E}[g|\mathcal{B}]$)

   $$\mathrm{E}[\phi(f)|\mathcal{B}] \geq a\,\mathrm{E}[f|\mathcal{B}] + b$$

   μ-a.e., i.e. on a set $E_{a,b}$ with null complement. (note: this only works if the space is finite, since a constant function is not L^1 if the space is not finite measure, therefore we had to make that assumption). Then taking the sup over $(a, b) \in A$ (A is countable), we have for almost every x (for all $x \in \cap_{(a,b) \in A} E_{a,b}$)

   $$\mathrm{E}[\phi(f)|\mathcal{B}](x) \geq \sup_{(a,b) \in A} a\,\mathrm{E}[f|\mathcal{B}](x) + b = \phi(\mathrm{E}[f|\mathcal{B}](x))$$

   which proves the claim. Applying this result with the convex function

   $$\phi(x) = |x|^p$$

   we have

   $$\mathrm{E}[|f|^p|\mathcal{B}](x) \geq |\,\mathrm{E}[f|\mathcal{B}](x)|^p$$

μ-a.e. Therefore

$$\int_X |\operatorname{E}[f|\mathcal{B}]|^p d\mu = \int_X \operatorname{E}[|f|^p|\mathcal{B}](x) d\mu = \int_X |f|^p d\mu$$

which is finite since $f \in L^p$. This proves the result for finite measure spaces. For the general case, it suffices to partition the space into countable many finite subsets, and apply the result on each:

$$X = \cup_n X_n$$

disjointly, then

$$\begin{aligned}\int_X |\operatorname{E}[f|\mathcal{B}]|^p d\mu &= \sum_n \int_{X_n} |\operatorname{E}[f|\mathcal{B}]|^p d\mu \\ &\leq \sum_n \int_{X_n} |f|^p d\mu \\ &= \int_X |f|^p d\mu \\ &= \|f\|_p^p\end{aligned}$$

which completes the proof.

3. If $g \in L^2$ is $\mathcal{B}$-measurable and $f \in L^1 \cap L^2$ is $\mathcal{A}$-measurable, then $\operatorname{E}[fg|\mathcal{B}] = g \operatorname{E}[f|\mathcal{B}]$ almost everywhere.

   proof without loss of generality, assume $f \geq 0$ and $g \geq 0$ (it suffices to prove the result for positive and negative parts of each function).

   If $g = 1_B$ for some $B \in \mathcal{B}$, then for all $E \in \mathcal{B}$,

   $$\int_E \operatorname{E}[fg|\mathcal{B}] = \int_E fg d\mu = \int_{E \cap B} f d\mu$$

   and since $E \cap B$ is in $\mathcal{B}$, by definition of $\operatorname{E}[f|\mathcal{B}]$,

   $$\int_{E \cap B} f d\mu = \int_{E \cap B} \operatorname{E}[f|\mathcal{B}] d\mu = \int_E g \operatorname{E}[f|\mathcal{B}] d\mu$$

   thus we have

   $$\int_E \operatorname{E}[fg|\mathcal{B}] d\mu = \int_E g \operatorname{E}[f|\mathcal{B}] d\mu$$

   for all $E \in \mathcal{B}$. By Lemma 1, $\operatorname{E}[fg|\mathcal{B}] = g \operatorname{E}[f|\mathcal{B}]$ μ-a.e.

   For simple functions, the result follows by linearity. Finally for the general case, the result follows by applying the monotone convergence theorem using a sequence of simple functions g_n that converge to g from below.

4. If $g \in L^2$ is $\mathcal{B}$-measurable and $f \in L^1 \cap L^2$ is $\mathcal{A}$-measurable, then $\int_X g \operatorname{E}[f|\mathcal{B}] d\mu = \int_X f \operatorname{E}[g|\mathcal{B}] d\mu$

   proof We have

   $$\begin{aligned}\int_X g \operatorname{E}[f|\mathcal{B}] d\mu &= \int_X \operatorname{E}[fg|\mathcal{B}] d\mu && \text{since } \operatorname{E}[gf|\mathcal{B}] = g \operatorname{E}[f|\mathcal{B}] \text{ a.e.} \\ &= \int_X fg d\mu && \text{by definition of the expectation} \\ &= \int_X f \operatorname{E}[g|\mathcal{B}] d\mu && \text{since } g \text{ and } \operatorname{E}[g|\mathcal{B}] \text{ are equal a.e., by Lemma 1}\end{aligned}$$

(6.4) Let $(X, \mathcal{A}, \mu)$ be a measure space. In the Hahn decomposition of a signed measure ν, X is decomposed as a disjoint union of sets $X = P \cup N$. These sets are not uniquely determined in general. Let $f \in L^1(X, \mathcal{A}, \mu)$ be real valued. Consider the measure $\nu(E) = \int_E f d\mu$ for all $E \in \mathcal{A}$. Describe in a concrete way, in terms of f, all subsets of X that arise as the set P in the Hahn decomposition of ν.

proof Let $P_0 = \{x : f(x) > 0\}$ and $E_0 = \{x : f(x) = 0\}$. Then the sets P of the Hahn decomposition are the sets of the form

$$P = (P_0 \cup E)\Delta Z$$

where $E \subseteq E_0$ and Z is a $\mu-$null set.

(6.5) Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $p, q \in (1, \infty)$ be conjugate exponents. Show that $L^q = (L^p)^*$.

proof Let $\ell \in (L^p)^*$ be a bounded linear functional. We seek a σ-finite subset $E \subset X$ such that ℓ is entirely determined by $\ell_{|L^p(E)}$, i.e. $\ell(f) = \ell(f|E)$. We construct this subset as follows:

by definition of the norm $\|\ell\| = \sup_{f \in L^p, \|f\|_p = 1} \ell(f)$, there exists, for all $n \in \mathbb{N}$, $f_n \in L^p$ such that $\|f_n\|_p = 1$ and $\|\ell\| - 1/n \leq \ell(f_n) \leq \|\ell\|$.

Now for all n, let $E_n = \{x \in X : |f_n(x)| > 0\}$. This is a σ-finite subset for the following reason: $E_n = \cup_{k \in \mathbb{N}} E_{n,k}$ where $E_{n,k} = \{x : |f_n(x)| \geq 1/k\}$, and $E_{n,k}$ has finite measure by the Chebyshev inequality $m(E_{n,k}) \leq k^p \|f_n\|_p^p$.

Let $E = \cup_{n \in \mathbb{N}} E_n$. E is σ-finite (countable union of $\sigma-$finite subsets). For all $g \in L^p$, we can decompose

$$\ell(g) = \ell_{|L^p(E)}(g_{|E}) + \ell_{|L^p(E^c)}(g_{|E^c})$$

since $g_{|E} \in L^p(E)$ and $g_{|E^c} \in L^p(E^c)$. Next, we show that $\ell_{|L^p(E^c)} = 0$, by showing that it has zero norm.

First, we show that

$$\|\ell\|^q \geq \|\ell_{|L^p(E)}\|^q + \|\ell_{|L^p(E^c)}\|^q \tag{1}$$

where q is conjugate to p, $1/p + 1/q = 1$. (in fact we can prove equality, but it is not needed here). For concision, let $\ell_E = \ell_{|L^p(E)}$ and $\ell_{E^c} = \ell_{|L^p(E^c)}$. Let $\epsilon > 0$. Then there exists $a \in L^p(E)$ and $b \in L^p(E^c)$ such that

$$\ell_E(a) \geq \|\ell_E\| - \epsilon$$
$$\ell_{E^c}(b) \geq \|\ell_{E^c}\| - \epsilon$$

now let $\alpha, \beta \geq 0$ to be determined, and consider the function $h = \alpha\tilde{a} + \beta\tilde{b}$ (where $\tilde{a}$ is a function on X that agrees with a on E and is zero on E^c). Then we have

$$\begin{aligned}
\|h\|_p^p &= \int_E |\alpha a|^p d\mu + \int_{E^c} |\beta b|^p d\mu \\
&= \alpha^p \int_E |a|^p d\mu + \beta^p \int_{E^c} |b|^p d\mu \\
&= \alpha^p \|a\|_p^p + \beta^p \|b\|_p^p \\
&= \alpha^p + \beta^p
\end{aligned}$$

in particular, $h \in L^p(X)$, thus

$$\begin{aligned}
\|\ell\| &\geq \frac{1}{\alpha^p + \beta^p} \ell(h) \\
&= \frac{1}{\alpha^p + \beta^p} (\alpha \ell_E(a) + \beta \ell_{E^c}(b)) \\
&\geq \frac{1}{\alpha^p + \beta^p} (\alpha \|\ell_E\| + \beta \|\ell_{E^c}\| - (\alpha + \beta)\epsilon)
\end{aligned}$$

now let

$$\alpha = \frac{\|\ell_E\|^{q-1}}{(\|\ell_E\|^q + \|\ell_{E^c}\|^q)^{1/p}} \qquad \beta = \frac{\|\ell_{E^c}\|^{q-1}}{(\|\ell_E\|^q + \|\ell_{E^c}\|^q)^{1/p}}$$

then we have

$$\alpha^p + \beta^p = \frac{\|\ell_E\|^{pq-p} + \|\ell_{E^c}\|^{pq-p}}{\|\ell_E\|^q + \|\ell_{E^c}\|^q} = 1$$

using $pq - p = q$. Observing that $\alpha + \beta \leq 2$ (each term is at most 1), the above inequality simplifies to

$$\begin{aligned} \|\ell\| &\geq \alpha\|\ell_E\| + \beta\|\ell_{E^c}\| - 2\epsilon \\ &= \frac{\|\ell_E\|^q + \|\ell_{E^c}\|^q}{(\|\ell_E\|^q + \|\ell_{E^c}\|^q)^{1/p}} - 2\epsilon \\ &= (\|\ell_E\|^q + \|\ell_{E^c}\|^q)^{1/q} - 2\epsilon \end{aligned}$$

using $1 - 1/p = 1/q$. Since this is true for arbitrary $\epsilon > 0$, we have

$$\|\ell\| \geq (\|\ell_E\|^q + \|\ell_{E^c}\|^q)^{1/q}$$

which proves the claim.

Now we have $\|\ell\|^q \geq \|\ell_{|L^p(E)}\|^q + \|\ell_{|L^p(E^c)}\|^q$. But

$$\begin{aligned} \|\ell\| &= \lim_n \ell(f_n) \\ &= \lim_n \ell_{|L^p(E)}(f_{n|E}) && \text{since } f_n \text{ is zero outside } E \\ &\leq \|\ell_{|L^p(E)}\| \end{aligned}$$

therefore

$$\|\ell_{|L^p(E^c)}\| \leq 0$$

i.e. equals zero. Therefore for all $f \in L^p(E^c)$, $\ell_{|L^p(E^c)}(f) = 0$ (otherwise the measure would be strict positive).

Finally, for all $h \in L^p(X)$, $\ell(h) = \ell_{|L^p(E)}(h_{|E})$ as claimed. Since E is σ-finite, there exists $g \in L^q(E)$ such that

$$\ell_{|L^p(E)}(h) = \int_E ghd\mu$$

for all $h \in L^p(E)$. Now extend g to be zero on E^c, and call the extension $\tilde{g}$. Then $\tilde{g} \in L^q(X)$, and we have for $h \in L^p(X)$,

$$\ell(h) = \ell_{|L^p(E)}(h_{|E}) = \int_E h_{|E}g = \int_X h\tilde{g}$$

since $\tilde{g}$ is zero on E^c. This shows surjectivity of the map

$$\begin{aligned} \phi : L^q &\to (L^p)^* \\ g &\mapsto h \mapsto \int_X ghd\mu \end{aligned}$$

Proof of injectivity is the same as in the σ-finite case: if $\int_X g_1 hd\mu = \int_X g_2 hd\mu$ for all $h \in L^p$, then $\int_X (g_1 - g_2)hd\mu = 0$, thus letting $E_k = \{x : g_1(x) - g_2(x) > 1/k\}$, E_k has finite measure by Chebyshev, thus $1_{E_k} \in L^p$, and $\int_{E_k}(g_1 - g_2)d\mu = 0$ i.e. $\mu(E_k) = 0$. Taking the union over k shows that $E = \{x : g_1(x) - g_2(x) \geq 0\}$ is a null set. By symmetry, we have the same for $F = \{x : g_2(x) - g_1(x) \geq 0\}$, which proves $g_1 = g_2$ a.e.

(6.6) If measures μ, ν on $\mathcal{A}$ are related by $\nu(E) = \int_E f d\mu$ for all $E \in \mathcal{A}$, then we write $f = \frac{d\nu}{d\mu}$ (f is unique up to redefinitions on μ-null sets). For $i \in \{1, 2\}$, Let $(X_i, \mathcal{A}_i, \mu_i)$ be σ-finite measure spaces. Let ν_i be measures on $(X_i, \mathcal{A}_i)$ satisfying $\nu_i \ll \mu_i$. Show that

1. $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$
2. for $\mu_1 \times \mu_2$ almost every $(x_1, x_2) \in X_1 \times X_2$,

$$\frac{d(\nu_1 \times \nu_2)}{\mu_1 \times \mu_2}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2)$$

proof

1. Let E be a $\mu_1 \times \mu_2$ null set. Then we have by Fubini's theorem

$$\begin{aligned} 0 &= (\mu_1 \times \mu_2)(E) \\ &= \int_{X_1} \int_{X_2} 1_E(x_1, x_2) d\mu_2 d\mu_1 \\ &= \int_{X_1} \mu_2(E^{x_1}) d\mu_1 \end{aligned}$$

we have $g : x_1 \mapsto \mu_2(E^{x_1})$ is a measurable non-negative function, and its integral is zero (i.e. $\|g\|_1 = 0$), thus the set $B = \{x_1 : \mu_2(E^{x_1}) > 0\}$ is a μ_1-null set, thus by absolute continuity $\nu_1 \ll \mu_1$ it is also a ν_1-null set. Therefore

$$\begin{aligned} \int_{X_1} \int_{X_2} 1_E(x_1, x_2) d\nu_2 d\nu_1 &= \int_{B^c} \int_{X_2} 1_E(x_1, x_2) d\nu_2 d\nu_1 \\ &= \int_{B^c} \nu_2(E^{x_1}) d\nu_1 \end{aligned}$$

now we observe that for all $x_1 \in B^c$, $\mu_2(E^{x_1}) = 0$, thus $\nu_2(E^{x_1}) = 0$ by absolute continuity. Therefore

$$\int_{X_1} \int_{X_2} 1_E(x_1, x_2) d\nu_2 d\nu_1 = 0$$

i.e. $\nu_1 \times \nu_2(E) = 0$.

2. Let $f_i = \frac{d\nu_i}{d\mu_i}$. We have $\tilde{f}_1 : (x_1, x_2) \mapsto f(x_1)$ is $\mathcal{A}_1 \times \mathcal{A}_2$ measurable since $\tilde{f}_1^{-1}(-\infty, c) = f_1^{-1}(-\infty, c) \times X_2$. Similarly $\tilde{f}_2 : (x_1, x_2) \mapsto f_2(x_2)$ is measurable, and so is the product $\tilde{f}_1\tilde{f}_2$. Therefore the set function

$$\begin{aligned} \lambda : \mathcal{A}_1 \times \mathcal{A}_2 &\to \mathbb{R} \\ E &\mapsto \int_E f_1(x_1) f_2(x_2) d\mu_1 \times \mu_2(x_1, x_2) \end{aligned}$$

is a measure. We show that $\lambda = \nu_1 \times \nu_2$. It suffices to show that they agree on measurable rectangles (then invoke a previous problem). Let $E_1 \in \mathcal{A}_1$ and $E_2 \in \mathcal{A}_2$. Then by Fubini's theorem, and using the fact that $1_{E_1 \times E_2}(x_1, x_2) = 1_{E_1}(x_1)1_{E_2}(x_2)$, we have

$$\begin{aligned} \lambda(E_1 \times E_2) &= \int_{X_1 \times X_2} f_1(x_1) f_2(x_2) 1_{E_1 \times E_2}(x_1, x_2) d\mu_1 \times \mu_2(x_1, x_2) \\ &= \int_{X_1} \int_{X_2} f_1(x_1) f_2(x_2) 1_{E_1}(x_1) 1_{E_2}(x_2) d\mu_2(x_2) d\mu_1(x_1) \\ &= \int_{X_1} f_1(x_1) 1_{E_1}(x_1) d\mu_1(x_1) \int_{X_2} f_2(x_2) 1_{E_2}(x_2) d\mu_2(x_2) \\ &= \nu_1(E_1)\nu_2(E_2) \end{aligned}$$

therefore (by problem 1.3) $\lambda = \nu_1 \times \nu_2$. Now let $g = \frac{d(\nu_1 \times \nu_2)}{\mu_1 \times \mu_2}(x_1, x_2)$. Then we have for all $E \in \mathcal{A}_1 \times \mathcal{A}_2$,

$$\int_E g d\mu_1 \times \mu_2 = (\nu_1 \times \nu_2)(E) = \lambda(E) = \int_E f_1(x_1) f_2(x_2) d\mu_1 \times \mu_2$$

therefore

$$\int_E (g(x_1, x_2) - f_1(x_1) f_2(x_2)) d\mu_1 \times \mu_2 = 0$$

on all measurable sets E. Using Lemma 1 proved earlier, we have $g(x_1, x_2) = f_1(x_1) f_2(x_2)$ for $\mu_1 \times \mu_2$ a.e. (x_1, x_2).

(6.7) Let $(X, \mathcal{A})$ be a set equipped with a $\sigma-$algebra of its subsets. Let $\mathcal{M}$ be the set of all signed measures on X. Observe that $\mathcal{M}$ is a vector space over $\mathbb{R}$, with operations $(\mu + \nu)(E) = \mu(E) + \nu(E)$ and $(t\mu)(E) = t\mu(E)$. Show that

1. $\|\mu\| = |\mu|(X)$ defines a norm on $\mathcal{M}$, where $|\mu| = \mu^+ + \mu^-$ by definition.
2. $\mathcal{M}$ is complete with respect to this norm

proof

1. By definition, we have for all $\mu \in \mathcal{M}$, $\|\mu\| = \mu^+(X) + \mu^-(X) \geq 0$.
   We have if $\|\mu\| = 0$ then $\mu^+(X) = \mu^-(X) = 0$, and for all $E \in \mathcal{A}$, $\mu^+(E) \leq \mu^+(X) = 0$, thus $\mu^+(E) = 0$. Similarly, $\mu^-(E) = 0$, therefore $\mu(E) = \mu^+(E) - \mu^-(E) = 0$, thus μ is identically zero on $\mathcal{A}$.
   Triangle inequality: let $\mu_1, \mu_2 \in \mathcal{M}$. We have

$$\begin{aligned}
\|\mu_1 + \mu_2\| &= (\mu_1 + \mu_2)^+(X) + (\mu_1 + \mu_2)^-(X) \\
&= \sup_{A \subset X} (\mu_1(A) + \mu_2(A)) + \sup_{A \subset X} (-\mu_1(A) - \mu_2(A)) \\
&\leq \sup_{A \subset X} \mu_1(A) + \sup_{A \subset X} \mu_2(A) + \sup_{A \subset X} -\mu_1(A) + \sup_{A \subset X} -\mu_2(A) \\
&= \mu_1^+(X) + \mu_2^+(X) + \mu_1^-(X) + \mu_2^-(X) \\
&= \|\mu_1\| + \|\mu_2\|
\end{aligned}$$

2. Let $(\mu)_n$ be a Cauchy sequence of elements of $\mathcal{M}$. Then we have $\forall \epsilon > 0$, there exists N such that for all $m, n \geq N$, $\|\mu_n - \mu_m\| \leq \epsilon$. For all $E \in \mathcal{A}$, we have $(\mu_n(E))_n$ is a Cauchy sequence. Define $\mu : \mathcal{A} \to \mathbb{R}$ by $\mu(E) = \lim_n \mu_n(E)$. Then μ is a signed measure, and $\lim_n \|\mu - \mu_n\| = 0$, therefore $(\mu_n)_n$ converges to $\mu \in \mathcal{M}$.