MATH 202B - Problem Set 5

Walid Krichene (23265217)

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(5.1) Show that there exists a continuous function $F : [0,1] \to \mathbb{R}$ which is monotonic on no interval of positive length.

proof We know there exists a continuous function $F : [0,1] \to \mathbb{R}$ that is nowhere differentiable. Then F is monotonic on no interval of positive length. Indeed, assume by contradiction that there exists an interval I with $m(I) > 0$ and F is monotonic on I. Then by Lebesgue's theorem, since F is monotone on I, F' exists almost everywhere on I. This contradicts the fact that F is nowhere differentiable.

(5.2) Construct an example of a nondecreasing function $F : \mathbb{R} \to \mathbb{R}$ such thar $F'(x)$ exists for every $x \in \mathbb{R}$, yet F' is not continuous.

proof Consider the function

$$\begin{aligned} F &: \mathbb{R} \to \mathbb{R} \\ x \mapsto F(x) &= \begin{cases} x & \text{if } x \leq 0 \\ x^2(\sin(1/x) + 1) + x & \text{if } x > 0 \end{cases} \end{aligned}$$

Continuity of f We have F is continuous on $(0, +\infty)$ and $(-\infty, 0)$. F is also left continuous at 0, and it remains to show right continuity at 0. We have for all $x > 0$, $0 \leq \sin(1/x) + 1 \leq 2$, thus

$$x \leq F(x) \leq 2x^2 + x$$

and both sides converge to 0 as x tends to 0. Therefore

$$\lim_{x \searrow 0} F(x) = 0 = F(0)$$

and F is continuous at 0.

Differentiability of f We have F is differentiable on $(-\infty, 0)$, and for all $x < 0$, $F'(x) = 1$. F is also differentiable on $(0, +\infty)$, and for all $x > 0$,

$$F'(x) = 2x(\sin(1/x) + 1) - \cos(1/x) + 1$$

and since $1 - \cos(1/x) \geq 0$ and $\sin(1/x) + 1 \geq 0$, then $F(x) \geq 0$ for all $x > 0$. For differentiability at 0, we have

$$\lim_{x \nearrow 0} \frac{F(x) - F(0)}{x - 0} = \lim_{x \nearrow 0} x \sin(1/x) + x + 1 = 1$$

since for all $x > 0$, $|x\sin(1/x)| \leq x$. We also have

$$\lim_{x \searrow 0} \frac{F(x) - F(0)}{x - 0} = \lim_{x \searrow 0} 1 = 1$$

therefore F is differentiable at 0 and $F'(0) = 1$.

Therefore F' exists everywhere, and satisfies $\forall x \in \mathbb{R}$, $F'(x) \geq 0$, and as a consequence, F is non decreasing on $\mathbb{R}$. However F' is not continuous at 0 since in the expression $F'(x) = 2x(\sin(1/x) + 1) - \cos(1/x) + 1$ all terms converge as $x \searrow 0$ except $\cos(1/x)$.

(5.3) Let f be a real valued function of bounded variation on an interval $[a, b]$. Show that if f is continuous at some $x \in [a, b]$, then $x \mapsto Vf([a, x])$ also continuous (where $V_f([a, x])$ is the total variation of f on $[a, x]$).

proof We overload the notation of the variation V_f, such that if P) is the partition $x_0 < x_1 < \cdots < x_n$, then

$$V_f(P) = \sum_{i=1}^{n} |f_i(x) - f_{i-1}(x)|$$

and the total variation of f on a segment is the supremum

$$V_f([a, b]) = \sup_{P \in P([a,b])} V_f(P)$$

First, we prove the following

Lemma 1 *If f is of bounded variation, and $\alpha < \beta < \gamma$, then*

$$V_f([\alpha, \gamma]) = V_f([\alpha, \beta]) + V_f([\beta, \gamma])$$

For any P partition of $[\alpha, \gamma]$, consisting of the points $\alpha = x_0 < \cdots < x_n = \gamma$, let x_{i_0} be such that $x_{i_0} \leq \beta < x_{i_0+1}$. Then we have

$$\begin{aligned} V_f(P) &= \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\ &\leq \left(\sum_{i=1}^{i_0} |f(x_i) - f(x_{i-1})| + |f(\beta) - f(x_{i_0})| \right) + \left(f(x_{i_0+1}) - f(\beta)| + \sum_{i=1}^{i_0+1} |f(x_i) - f(x_{i-1})| \right) \\ &\leq V_f([\alpha, \beta]) + V_f([\beta, \gamma]) \end{aligned}$$

thus taking the sup over all such partitions, we have

$$V_f([\alpha, \gamma]) \leq V_f([\alpha, \beta]) + V_f([\beta, \gamma])$$

To show the reverse inequality, we have for all $\epsilon > 0$, there exist P and P', partitions of $[\alpha, \beta]$ and $[\beta, \gamma]$ respectively, such that

$$\begin{aligned} V_f(P) &> V_f([\alpha, \beta]) - \epsilon/2 \\ V_f(P') &> V_f([\beta, \gamma]) - \epsilon/2 \end{aligned}$$

now consider $P \cup P'$. This is a partition of $[\alpha, \gamma]$, and

$$\begin{aligned} V_f([\alpha, \gamma]) &\geq V_f(P \cup P') \\ &\geq V_f(P) + V_f(P') \\ &\geq V_f([\alpha, \beta]) + V_f([\beta, \gamma]) - \epsilon \end{aligned}$$

since this holds for all ϵ, we have $V_f([\alpha, \gamma]) \geq V_f([\alpha, \beta]) + V_f([\beta, \gamma])$ and the Lemma is proved.

proof Let $x \in [a,b]$ be fixed, and let f be continuous at x. We first show that $V(f)$ is right continuous at x. Let $\epsilon > 0$. We seek to find $\delta > 0$ such that for all $y \in [a,b]$, $|y - x| \leq \delta \Rightarrow |V(f)(x) - V(f)(y)| \leq \epsilon$.

By definition of the total variation of f on $[x,b]$ (which is finite since f is of bounded variation), there exists a partition $P_{[x,b]}$ of $[x,b]$ such that

$$V_f(P_{[x,b]}) \geq V_f([x,b]) - \epsilon/2$$

let x_1 be the smallest point in $P \setminus \{x\}$. By continuity of f at x, there exists $\delta > 0$ such that if $|y - x| \leq \delta$, then $|f(x) - f(y)| \leq \epsilon/2$. Let $\delta' = \min(\delta, x_1 - x)$. Let $y \in (x, x + \delta')$ (this ensures that $x \leq y < x_1$). Now consider the partition $[x,b]$ obtained by adding y to P

$$P'_{[x,b]} = P_{[x,b]} \cup \{y\}$$

By the triangle inequality, we have

$$V_f(P'_{[x,b]}) \geq V_f(P_{[x,b]})$$

Finally, consider the partition of $[y,b]$, obtained by removing x from $P'[x,b]$

$$P_{[y,b]} = P'_{[x,b]} \setminus \{x\}$$

we have

$$V_f(P'_{[x,b]}) = |f(x) - f(y)| + V_f(P_{[y,b]}) \leq \epsilon/2 + V_f(P_{[y,b]})$$

Combining these inequalities, we have

$$\begin{aligned}
V_f([x,y]) &= V_f([x,b]) - V_f([y,b]) && \text{by the Lemma} \\
&\leq V_f(P_{[x,b]}) + \epsilon/2 - V_f([y,b]) \\
&\leq V_f(P'_{[x,b]}) + \epsilon/2 - V_f([y,b]) \\
&\leq \epsilon/2 + V_f(P_{[y,b]}) + \epsilon/2 - V_f([y,b]) \\
&\leq \epsilon
\end{aligned}$$

Therefore $\forall y \in (x, x + \delta')$, $V_f([x,y]) \leq \epsilon$. Therefore $x \mapsto V_f([a,x])$ is right continuous at x. A similar argument shows that it is left continuous at x, hence continuity.

(5.4) Let $K \subset \mathbb{R}$ be a countable compact set. Show that there exists a Borel measure μ such that $\mu(K) = 1$, $\mu(\mathbb{R} \setminus K) = 0$, and $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$.

proof If there exists an open interval I such that $I \subseteq K$ and $\mu(I) > 0$, then the measure μ defined by

$$\mu(E) = \frac{1}{m(I)} m(E \cap I) = \frac{1}{m(I)} \int_E 1_I dm$$

satisfies the desired properties, since $\mu(K) = \frac{1}{\mu(I)}\mu(I) = 1$, $\mu(\mathbb{R} \setminus K) = \frac{1}{\mu(I)}\mu(\emptyset) = 0$, and for all $x \in \mathbb{R}$, $\mu(\{x\}) = \frac{1}{\mu(I)} m(\{x\} \cap I) = 0$.

Now consider the case where there exists no such interval, i.e. $\text{int}(K) = \emptyset$. K^c is open, thus is a countable union of disjoint open interval.

$$K = \cup_{n \in \mathbb{N}} I_n$$

where each $I_n = (a_n, b_n)$.

Inspired by the construction of the Cantor function, we seek to construct a sequence of functions F_n that are continuous non decreasing, and that satisfy:

$$\forall m \geq n, \ |F_n - F_m| \leq 2^{-n}$$

and such that their limit is constant on K^c. We organize the construction in "generations", where in each generation n, we have $2^n + 1$ intervals $(I_{n,k})$, $k \in \{0, \dots, 2^n\}$, and such that

- $I_{n,k}$ is always (strictly) to the left of $I_{n,k+1}$, i.e. $b_{n,k} < a_{n,k+1}$
- the set of points in K between any two consecutive intervals is uncountable, i.e. for all k, $(b_{n,k}, a_{n,k+1}) \cap K$ is uncountable.

More precisely, for $n = 0$, we observe that since K is bounded, there exists n_0 such that $a_{n_0} = -\infty$ and n_1 such that $b_{n_1} = +\infty$.

$$I_{0,0} = I_{n_0}$$
$$I_{0,1} = I_{n_1}$$

then we have the desired properties. Then by induction, assume $\{I_{n,k}\}_k$ constructed. Then for all even k, define $I_{n+1,k/2} = I_{n,k}$, and for all odd k, find an interval (a_m, b_m) such that $b_{n,k/2} < a_m < b_m < a_{n,k/2+1}$ such that $(b_{n,k/2} < a_m) \cap K$ and $(b_{n,k/2} < a_m) \cap K$ are both uncountable. The following Lemma guarantees that we can do this construction

Lemma 2 *If $J \subseteq [\alpha, \beta]$ is an uncountable compact set with empty interior, then there exists an open interval $(a, b) \subset J^c$ such that $[\alpha, a] \cap J$ and $[b, \beta] \cap J$ are both uncountable.*

The complement of J is open, thus is the countable union of open intervals $J^c = \cup_{n \in \mathbb{N}} I_n$. By contradiction, assume that for all n, one of $[\alpha, a_n] \cap J$ and $[b_n, \beta] \cap J$ is countable. Call it C_n. $C = \cup_{n \in \mathbb{N}} C_n$ is a countable subset of J, thus $J \setminus C$ is uncountable. Let $x < y$ be two points in $J \setminus C$. Since J has empty interior, there exists $z \in (x, y) \cap J^c$, in particular there exists n such that $z \in I_n$, and $I_n \subseteq [x, y]$. Therefore x and y cannot be on the same side of I_n, and one of them has to be in C_n. This contradicts the definition of x, y. This concludes the proof of the lemma.

Now that the intervals $I_{n,k}$ are constructed, we can define $F_n : \mathbb{R} \to [0, 1]$ by

$$F_n(x) = k/2^n \forall x \in I_{k,n}$$

and F_n is affine between $b_{n,k}$ and $a_{n,k+1}$, i.e. $F_n(x) = 2^{-n}(k + \frac{x - b_{n,k}}{a_{n,k} - b_{n,k}})$. By construction, F_n is continuous non decreasing. We also have that for all $m \geq n$, F_m agrees with F_n on $\cup_k I_{n,k}$, and for all $x \notin \cup_k I_{n,k}$ there exists k such that $x \in [b_{n,k}, a_{n,k+1}]$, then both $F_n(x)$ and $F_m(x)$ are in $[k/2^n, (k+1)/2^n]$, therefore

$$\sup_x |F_n(x) - F_m(x)| \leq 2^{-n+1}$$

and $(F_n)_n$ is a Cauchy sequence (for the sup). Then it converges and its limit F is continuous (for every x, $(F_n(x))_n$ is Cauchy, thus it converges. Let $F(x)$ be its limit. Now $(F - F_n)$ converges uniformly to zero, therefore F is the uniform limit of continuous functions and is continuous).

F is also nondecreasing as the limit of a sequence of nondecreasing functions. Finally, F is constant on each interval $I_n \subseteq K^c$. Indeed, let $x < y \in I_n$, and let $\epsilon > 0$. We show that $|F(x) - F(y)| \leq \epsilon$. Let m such that $2^{-m} \leq \epsilon$, and consider the intervals $I_{m,k}$, $k \in \{0, \ldots, 2^m\}$. If I_n is one of these intervals, then we are done (F is by definition constant on such intervals). Otherwise, there exists k such that $b_{m,k} < x < y < b_{m,k}$ (the intervals that form K^c are disjoint). Therefore $k/2^n \leq F(x) \leq F(y) \leq (k+1)/2^n$, which proves that

$$|F(y) - F(x)| \leq 2^{-n} \leq \epsilon$$

since this holds for arbitrary ϵ, we have equality.

(5.5) Find all continuous nondecreasing functions $F : \mathbb{R} \to \mathbb{R}$ such that $F'(x) = 0$ for all but countably many $x \in \mathbb{R}$

proof Any such function is constant. To prove this, let $a < b \in \mathbb{R}$. We seek to prove that $F(a) = F(b)$. Let $E = \{x \in \mathbb{R} | f'(x) = 0\}$. The complement of E is countable by assumption, so let $E^c = \{b_n, n \in \mathbb{N}\}$ be the set of "bad" points.

Now let $\epsilon > 0$. Let $x \in E$. We have $f'(x) = 0$, i.e.

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = 0$$

Thus there exists δ_x such that

$$\forall y \in B(x, \delta_x),\ |f(y) - f(x)| \leq \epsilon |y - x|$$

For all $b_n \in E^c$, by continuity of f at b_n, there exists $\delta_{b_n} > 0$ such that

$$\forall y \in B(b_n, \delta_{b_n}),\ |f(b_n) - f(y)| \leq \epsilon / 2^n$$

The open balls $\{B(x, \delta_x), x \in E\} \cup \{B(b_n, \delta_{b_n}), n \in \mathbb{N}\}$ form an open cover of $\mathbb{R}$ (indeed, $\mathbb{R} = E \cup E^c$, and for all $x \in E$, $x \in B(x, \delta_x)$, and for all $b_n \in E^c$, $b_n \in B(b_n, \delta_{b_n})$), so in particular, it forms an open cover of $[a, b]$. Since $[a, b]$ is compact, we can extract a finite subcover, thus there exists a finite set of points $a \leq x_1 \leq \cdots \leq x_K \leq b$ such that $[a, b] \subseteq \cup_{k=1}^{K} B(x_k, \delta_{x_k})$. Denote by $B_k = B(x_k, \delta_{x_k})$.

We assume, without loss of generality, that the set $\{B_1, \ldots, B_K\}$ is minimal (for inclusion). In this case, we have for all k, $B_k \cap B_{k+1} \neq \emptyset$ [1]. For all $k \in \{1, \ldots, K-1\}$, let $p_k \in B_k \cap B_{k+1}$, and let $p_0 = a$ and $p_K = b$. This defines a partition

$$a = p_0 \leq p_1 \leq \cdots \leq p_K = b$$

such that for all $k \in \{1, \ldots, K\}$, p_{k-1} and p_k are contained in B_k. Therefore we have

$$F(b) - F(a) = \sum_{k=1}^{K} |F(p_k) - F(p_{k-1})|$$

and for all k, $|F(p_k) - F(p_{k-1})| \leq |F(p_k) - F(x_k)| + |F(x_k) - F(p_{k-1})|$, two cases can occur

- if x_k is a good point (i.e. $x_k \in E$), then

$$|F(p_k) - F(x_k)| + |F(x_k) - F(p_{k-1})| \leq \epsilon |p_k - x_k| + \epsilon |x_k - p_{k-1}| = \epsilon |p_k - p_{k-1}|$$

- if x_k is a bad point, i.e. $x_k = b_n$ for some n, then

$$|F(p_k) - F(x_k)| + |F(x_k) - F(p_{k-1})| \leq 2\epsilon / 2^n$$

therefore

$$\begin{aligned} F(b) - F(a) &= \sum_{k=1}^{K} |F(p_k) - F(p_{k-1})| \\ &\leq \sum_{k=1}^{K} \epsilon |p_k - p_{k-1}| + \sum_{n \in \mathbb{N}} 2\epsilon / 2^n \\ &\leq \epsilon |b - 1| + 4\epsilon \end{aligned}$$

since this is true for arbitrary $\epsilon > 0$, we have $F(b) - F(a) = 0$.

[1] if $B_k \cap B_{k+1} = \emptyset$, then $x_k + \delta_{x_k} \leq x_{k+1} - \delta_{x_{k+1}}$, and there exists p such that

$$x_k + \delta_{x_k} \leq p \leq x_{k+1} - \delta_{x_{k+1}}$$

i.e. p is not in $B_k \cup B_{k+1}$. Since we have a cover, there exists k' such that $p \in B_{k'}$, but then we either have $k' > k + 1$, in which case $B_{k'} \supset B_{k+1}$, or $k' < k$, in which case $B_{k'} \supset B_k$. Both cases contradict minimality of the cover.

(5.6.a) Show that if $p \in [1, \infty)$ and $f \in L^p(\mathbb{R}^n)$ with respect to the Lebesgue measure, then the mapping

$$\begin{aligned} \phi : \mathbb{R}^n &\to L^p \\ t &\mapsto \phi(t) = f_t = x \mapsto f(x-t) \end{aligned}$$

is continuous. In particular, $\lim_{t \to 0} f_t = f$ in L^p norm. Here L^p is the metric space of equivalence classes.

proof Let $\epsilon > 0$. We have the space of continuous compactly supported functions (C_0) is dense in L^p. Therefore there exists a function $g \in C_0$ such that

$$\|f - g\|_p \leq \epsilon/3$$

We also have for all t, $\|f_t - g_t\|_p \leq \epsilon/3$ since $\|f_t - g_t\|_p^p = \int_{\mathbb{R}^n} |f(x-t) - g(x-t)|^p dx = \int_{\mathbb{R}^n} |f(y) - g(y)|^p dy$ by the change of variable $y = x - t$. Now we have by the triangle inequality, for all t,

$$\begin{aligned} \|f - f_t\|_p &\leq \|f - g\|_p + \|g - g_t\|_p + \|g_t - f_t\|_p \\ &\leq \epsilon/3 + \|g - g_t\|_p + \epsilon/3 \end{aligned}$$

in order to bound the term $\|g - g_t\|_p$, we use the fact that there exists a compact $K \subset \mathbb{R}^n$ such that g is zero outside of K. Then let $K' = K + B(0,1) = \{x + y | x \in K, \ |y| \leq 1\}$. Then we have for all $|t| \leq 1$, g_t is zero outside K' ($g_t(x) \neq 0 \Rightarrow g(x-t) \neq 0 \Rightarrow x - t \in K \Rightarrow x \in K'$). Therefore for all $|t| \leq 1$, $g - g_t$ is zero outside of K', and

$$\|g - g_t\|_p^p = \int_{K'} |g - g_t|^p dm$$

g is continuous on the compact set $K' + B(0,1)$, then g is uniformly continuous on that set, and there exists $\delta > 0$ (and $\delta < 1$) such that for all $x \in K'$ and for all $t \in B(0, \delta)$, $|g(x) - g(x-t)| \leq \frac{\epsilon}{3(1+m(K'))}$.

Then for all $t \in B(0, \delta)$, integrating over K', we have

$$\|g - g_t\|_p \leq m(K') \frac{\epsilon}{3(1 + m(K'))} \leq \epsilon/3$$

therefore for all $t \in B(0, \delta)$

$$\|f - f_t\|_p \leq 3\epsilon/3$$

which concludes the proof.

(5.6.b) The previous result is false for $p = \infty$. However, show that if $f \in C_0(\mathbb{R}^n)$ (continuous functions with a bounded support), then

$$\lim_{t \to 0} \|f_t - f\|_\infty = 0$$

proof Let $f \in C_0$. Then there exists a compact $K \subset \mathbb{R}^n$ such that f is zero outside K. As previously, consider $K' = K + B(0,1)$. We have for all $t \in B(0,1)$, both f and f_t are zero outside K'

Since f is continuous on the compact $K' + B(0,1)$, it is uniformly continuous. Let $\epsilon > 0$. Then there exists $\delta > 0$ (and $\delta < 1$) such that for all $x, y \in K' + B(0,1)$, $|x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \epsilon$. Then for all $t \in B(0, \delta)$, we have for all $x \in K'$, both x and $x - t$ are in $K' + B(0,1)$, and $|x - (x-t)| \leq \delta$, thus $|f(x) - f_t(x)| \leq \epsilon$.

Therefore for all $t \in B(0, \delta)$, we have for all $x \notin K'$, $f(x) = f_t(x) = 0$, and for all $x \in K'$, $|f(x) - f_t(x)| \leq \epsilon$, therefore

$$\|f - f_t\|_\infty \leq \sup_{x \in \mathbb{R}^n} |f(x) - f_t(x)| \leq \epsilon$$

this concludes the proof.

(5.6.c) Suppose that $f \in L^\infty$, and that $\|f_t - f\|_\infty \to 0$. Must f agree almost everywhere with a continuous function?