MATH 202B - Problem Set 4

Walid Krichene (23265217)

February 27, 2013

(4.1) Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial which does not vanish identically. Use Fubini's Theorem to show, by induction on the dimension n, that $Z(P) = \{x \in \mathbb{R}^n | P(x) = 0\}$ has Lebesgue measure equal to zero.

proof By induction on n. For $n = 1$, consider a polynomial $P \in \mathbb{R}[X]$, and let $d = \deg P$. Then P has at most d roots, and $Z(P) = \{x \in \mathbb{R} | P(x) = 0\}$ is finite, thus has measure 0.

Now let $n \geq 1$, and assume the result is true for n. Consider a polynomial $P \in \mathbb{R}[X_1, \dots, X_{n+1}]$ and assume that P is not identically zero. Viewing P as a polynomial in $\mathbb{R}[X_{n+1}][X_1, \dots, X_n]$ (i.e. a polynomial with coefficients in $\mathbb{R}[X_{n+1}]$), we have that there exists A a finite subset of $\mathbb{N}^n$, and a collection of polynomials $Q_\alpha \in \mathbb{R}[X_1]$, $\alpha \in A$ such that

$$P(X_1, \dots, X_{n+1}) = \sum_{\alpha \in A} Q_\alpha(X_{n+1}) X_{1:n}^\alpha$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, and $X_{1:n}^\alpha = \prod_{i=1}^n X_i^{\alpha_i}$.

Since P is non identically zero, then there exists $\alpha_0 \in A$ such that $Q_{\alpha_0} \not\equiv 0$ (otherwise, $\forall x_1 \in \mathbb{R}$ and $\forall \alpha \in A$, $Q_\alpha(x_{n+1}) = 0$, thus $\forall (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$, $P(x_1, \dots, x_n) = 0$). Let

$$\begin{aligned} Z(Q_{\alpha_0}) &= \{x_{n+1} \in \mathbb{R} | Q_{\alpha_0}(x_{n+1}) = 0\} \\ Z(P) &= \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} | P(x_1, \dots, x_{n+1}) = 0\} \end{aligned}$$

and for all $x_{n+1} \in \mathbb{R}$,

$$Z(P(\cdot, x_{n+1})) = \{(x_1, \dots, x_n) \in \mathbb{R}^n | P(x_1, \dots, x_{n+1}) = 0\}$$

Since $Q_{\alpha_0} \not\equiv 0$, $Z(Q_{\alpha_0})$ is finite thus has zero measure (using the case $n = 1$). Therefore

$$\begin{aligned} m(Z_P) &= \int_{\mathbb{R}^{n+1}} 1_{Z(P)} dm \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} 1_{Z(P)} dm(x_{n+1}) \right) dm(x_1, \dots, x_n) && \text{by Fubini's thm} \\ &= \int_{\mathbb{R}^n} \left(\int_{Z(Q_{\alpha_0})^c} 1_{Z(P)} dm(x_{n+1}) \right) dm(x_1, \dots, x_n) && \text{since } Z(Q_{\alpha_0}) \text{ has zero measure} \\ &= \int_{Z(Q_{\alpha_0})^c} \left(\int_{\mathbb{R}^n} 1_{Z(P)} dm(x_1, \dots, x_n) \right) dm(x_{n+1}) && \text{by Fubini's thm} \\ &= \int_{Z(Q_{\alpha_0})^c} m(Z(P(., x_{n+1}))) dm(x_{n+1}) \end{aligned}$$

now for all $x_{n+1} \in Z(Q_{\alpha_0})^c$, $P(X_1, \dots, X_n, x_{n+1})$ (viewed as an element of $\mathbb{R}[X_1, \dots, X_n]$) is not identically zero (since $Q_{\alpha_0}(x_{n+1}) \neq 0$). Therefore by the induction hypothesis, $Z(P(., x_{n+1}))$ has zero measure. Therefore

$$m(Z_P) = \int_{Z(Q_{\alpha_0})^c} 0 dm(x_{n+1}) = 0$$

(4.2) The gamma function is

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

defined initially for all $z \in \mathbb{C}$ with real part greater than 0. Use Fubini's Theorem to prove the identity

$$\int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for all $x, y \in \mathbb{C}$ with strictly positive real parts.

proof

$$\begin{aligned}(1)\ \Gamma(x+y)\int_0^1 t^{x-1}(1-t)^{y-1}dt &= \int_{[0,\infty)} e^{-s}s^{x+y-1}ds \int_{[0,1]} t^{x-1}(1-t)^{y-1}dt \\ &= \int_{[0,\infty)}\left(\int_{[0,1]} e^{-s}(st)^{x-1}s^y(1-t)^{y-1}dt\right)ds\end{aligned}$$

For all $s \neq 0$, using the change of variable $u = \phi^{-1}(t)$ where

$$\begin{aligned}\phi : [0,s] &\to [0,1] \\ u &\mapsto \phi(u) = u/s\end{aligned}$$

we have $|J_\phi| = s^{-1}$, and $u = ts$, thus

$$\int_{[0,1]} e^{-s}(st)^{x-1}s^y(1-t)^{y-1}dt = \int_{[0,s]} e^{-s}u^{x-1}s^y(1-u/s)^{y-1}s^{-1}du = \int_{[0,s]} e^{-s}u^{x-1}(s-u)^{y-1}du$$

and we observe that the inequality also holds for $s = 0$ (both integrals are zero). Thus we have

$$\begin{aligned}(1) &= \int_{[0,\infty)}\left(\int_{[0,s]} e^{-s}u^{x-1}(s-u)^{y-1}du\right)ds \\ &= \int_D e^{-s}u^{x-1}(s-u)^{y-1}d(u,s) \qquad \text{where } D = \{(s,u) \in \mathbb{C}\times\mathbb{C} | u \leq s\} \\ &= \int_{[0,\infty)}\left(\int_{[u,\infty)} e^{-s}u^{x-1}(s-u)^{y-1}ds\right)du\end{aligned}$$

next, using the change of variable $z = \phi^{-1}(s)$ where

$$\begin{aligned}\phi : [0,\infty) &\to [u,\infty) \\ z &\mapsto \phi(z) = z + u\end{aligned}$$

we have $|J_\phi(z)| = 1$ and $z = s - u$, thus

$$\int_{[u,\infty)} e^{-s}u^{x-1}(s-u)^{y-1}ds = \int_{[0,\infty)} e^{-z-u}u^{x-1}z^{y-1}dz$$

therefore

$$\begin{aligned}(1) &= \int_{[0,\infty)}\left(\int_{[0,\infty)} e^{-z-u}u^{x-1}z^{y-1}dz\right)du \\ &= \Gamma(x)\Gamma(y)\end{aligned}$$

(4.3) Let μ be a finite Borel measure on $\mathbb{R}^n$, Define

$$f(x) = \int_{\mathbb{R}^n} \|x-y\|^{-r} d\mu(y)$$

where $r \in (0, n)$ is fixed. Prove that $f(x) < \infty$ for almost every $x \in \mathbb{R}^n$, with respect to Lebesgue measure.

proof Let

$$E = \{x \in \mathbb{R}^n | f(x) = \infty\}$$

Consider the function

$$\begin{aligned} h : \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R} \cup \{\infty\} \\ (x, y) &\mapsto h(x, y) = \|x-y\|^{-r} \end{aligned}$$

h is measurable wrt $\mathcal{B}_{2n}$[1].

Now let $K = \mathrm{cl}(B(0, \eta))$ be a closed ball of $\mathbb{R}^n$. Then we have by Fubini's theorem

$$\int_{K \times \mathbb{R}^n} h(x, y) d(m \times \mu)(x, y) = \int_K \left(\int_{\mathbb{R}^n} h(x, y) d\mu(y) \right) dm(x) = \int_{\mathbb{R}^n} \left(\int_K h(x, y) dm(x) \right) d\mu(y)$$

Letting $g_K(y) = \int_K h(x, y) dm(x)$, we have

$$\int_K f(x) dm(x) = \int_{\mathbb{R}^n} g_K(y) d\mu(y)$$

Next, we show that g_K is a bounded function. We have

$$g_K(y) = \int_K \|x-y\| dm(x) = \int_{K-y} \|z\| dm(z)$$

using the change of variable $z = \phi^{-1}(x) = x - y$, with $J_\phi = 1$. Now we consider two cases (y close to zero and y far from zero)

- let $y \in B(0, \eta + 1)$, then

  $$K - y \subset B(0, 2\eta + 1)$$

  (indeed, if $z \in K - y$, then $\|z\| \le \|z - y\| + \|y\| < \eta + (\eta + 1)$) therefore

  $$\begin{aligned} g_K(y) &= \int_{K-y} \|z\|^{-r} dm(z) \\ &\le \int_{B(0,2\eta+1)} \|z\|^{-r} dm(z) \end{aligned}$$

  then using a radial change of variable $(\rho, \theta, \psi_1, \ldots, \psi_{n-2}) = \phi^{-1}(z)$, where

  $$\begin{aligned} \phi : (0, \infty) \times [0, 2\pi) \times [0, \pi)^{n-2} &\to \mathbb{R}^n \\ (\rho, \theta, \psi_1, \ldots, \psi_{n-2}) &\mapsto \rho u(\theta, \psi_1, \ldots, \psi_{n-2}) \end{aligned}$$

---

[1] Let $\Delta = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$. Delta is measurable since it is the inverse image of 0 by the continuous function $(x, y) \mapsto (x - y)$. Then we can write h as the sum of measurable functions

$$h = \bar{h} 1_{\Delta^c} + \infty 1_\Delta$$

where

$$\bar{h}(x, y) = \begin{cases} |x-y|^{-r} & \text{if } (x, y \in \Delta) \\ 0 & \text{otherwise} \end{cases}$$

thus h is measurable

where $u(\theta, \psi_1, \dots, \psi_{n-2})$ is a unit vector in $\mathbb{R}^n$. We have $J_\phi(\rho, \theta, \psi_1, \dots, \psi_{n-2}) = \rho^{n-1}\gamma(\theta, \psi_1, \dots, \psi_{n-2})$, and $\|z\| = \rho$. Thus

$$\int_{B(0,2\eta+1)} \|u\|^{-r} dm(u) = C \int_{(0,2\eta+1]} \rho^{-r}\rho^{n-1} dm(\rho)$$

for some positive constant C (here we have integrated over $\theta, \psi_1, \dots, \psi_{n-2}$). And since $n - 1 - r > -1$

$$\int_{(0,2\eta+1]} \rho^{n-1-r} dm(\rho) = \left[\frac{1}{n-r}\rho^{n-r}\right]_0^{2\eta+1} < \infty$$

denoting M the finite value of the integral, we have

$$g_K(y) \leq CM$$

- let $y \notin B(0, \eta + 1)$, then by the triangle inequality, for all $z \in K = \text{cl}(B(0, \eta))$,

  $$\|x - y\| \geq \|y\| - \|x\| \geq \eta + 1 - \eta = 1$$

  and since $r > 0$, then $\|x - y\|^{-r} \leq 1$. Thus

  $$\begin{aligned} g(y) &= \int_K \|x - y\| dm(x) \\ &\leq \int_K 1 dm(x) \\ &= m(K) \end{aligned}$$

Therefore for all y, $g(y) \leq \max(CM, m(K)) < \infty$. Finally, since μ is finite and g_K is bounded, we have $\int_{\mathbb{R}^n} g_K(y) d\mu(y) < \infty$, i.e. $\int_K f(x) dm(x) < \infty$, thus $E \cap K$ has zero measure, since

$$\int_{\mathbb{R}^n} f(x) dm(x) \geq \int_{E\cap K} f(x) dm(x) = \infty m(E \cap K)$$

Finally, $(E \cap \text{cl}(B(0, n)))_n$ is a sequence of increasing nested sets that converges to E, therefore

$$m(E) = \lim_n m(E \cap \text{cl}(B(0, n))) = 0$$

since each term in the sequence is a null set.

(4.4) Let $\mathcal{C} \subset [0,1]$ be the standard Cantor set (defined by deleting middle thirds of intervals). Recall that $C = \cap_{n=1}^{\infty} \mathcal{C}_n$ where $\mathcal{C}_n$ is a certain union of 2^n intervals $I_{k,n}$, each having length 3^{-n}. Let F be the associated Cantor-Lebesgue function (constructed in our text and in class). Let μ be the Stieltjes measure associated to F. Show that $\mu(I_{k,n}) = 2^{-n}$ for all k, n

proof For all n, and for all $k \in \{1, \ldots, 2^n\}$, let $I_{k,n} = [a_{k,n}, b_{k,n}]$. We have by definition

$$\forall x \in (b_{k,n}, a_{k+1,n}),\ F(x) = k/2^n$$
$$\forall x \in (-\infty, 0),\ F(x) = 0$$
$$\forall x \in (1, +\infty),\ F(x) = 1$$

And since F is continuous, we have

$$\forall k \geq 2,\ F(a_{k,n}) = \lim_{x \nearrow a_{k,n}} F(x) = (k-1)/2^n \quad \text{since for all } x \in (b_{k-1,n}, a_{k,n}),\ F(x) = (k-1)/2^n$$
$$\forall k \leq 2^n - 1,\ F(b_{k,n}) = \lim_{x \searrow b_{k,n}} F(x) = k/2^n \quad \text{since for all } x \in (b_{k,n}, a_{k+1,n}),\ F(x) = k/2^n$$

the above equalities also hold for $k = 1$ and $k = 2^n$, since we have $a_{1,n} = 0$ and $b_{2^n,n} = 1$, and

$$F(0) = \lim_{x \nearrow 0} F(x) = 0$$
$$F(1) = \lim_{x \searrow 1} F(x) = 1$$

Finally, since F is continuous, we have $\mu(I_{k,n}) = F(b_{k,n}) - F(a_{k,n}) = k/2^n - (k-1)/2^n = 1/2^n$.

(4.5) Is it possible for a Lebesgue measurable set $E \subset \mathbb{R}$ to satisfy $.001m(I) \leq m(E \cap I) \leq .999m(I)$ for every bounded interval $I \subset \mathbb{R}$? Justify.

proof Let E be a measurable set. We show that the above property cannot be satisfied.

Consider the indicator function $f = 1_E$. We have f is measurable, and integrable on every bounded interval I, since $\int_I 1_E dm = m(E \cap I) \leq m(I)$. Thus we have by Corollary 7.4, the function

$$\begin{aligned} F &: \mathbb{R} \to \mathbb{R} \\ x &\mapsto F(x) = \int_{(0,x]} 1_E dm \end{aligned}$$

is differentiable almost everywhere, and for almost every x,

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = 1_E(x)$$

We consider two cases:

1. If $E = \emptyset$, then we must have $m(E \cap I) = 0$ for all bounded interval I.
2. If $E \neq \emptyset$, then let $x \in E$. We have

   $$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = 1_E(x) = 1$$

   Taking $\epsilon = 0.001$, we have: there exists $h > 0$ such that $\frac{F(x+h)-F(x)}{h} > 1 - \epsilon = 0.999$, which we can rewrite as

   $$\int_{(x,x+h]} 1_E dm > 0.999h$$

   letting $I = (x, x+h]$, this is equivalent to

   $$m(E \cap I) > 0.999m(I)$$

   therefore there exists a bounded interval I for which the property is not satisfied.

(4.6) Find all continuous nondecreasing functions $F : \mathbb{R} \to \mathbb{R}$ which satisfy $F'(x) = 0$ for all but at most countably many $x \in \mathbb{R}$. Justify.

proof Any such function is constant. To prove this, let $a < b \in \mathbb{R}$. We seek to prove that $F(a) = F(b)$. Let $E = \{x \in \mathbb{R} | f'(x) = 0\}$. The complement of E is countable by assumption, so let $E^c = \{b_n, n \in \mathbb{N}\}$ be the set of "bad" points.

Now let $\epsilon > 0$. Let $x \in E$. We have $f'(x) = 0$, i.e.

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = 0$$

Thus there exists δ_x such that

$$\forall y \in B(x, \delta_x),\ |f(y) - f(x)| \le \epsilon |y - x|$$

For all $b_n \in E^c$, by continuity of f at b_n, there exists $\delta_{b_n} > 0$ such that

$$\forall y \in B(b_n, \delta_{b_n}),\ |f(b_n) - f(y)| \le \epsilon / 2^n$$

Finally, the open balls $\{B(x, \delta_x), x \in E\} \cup \{B(b_n, \delta_{b_n}), n \in \mathbb{N}\}$ form an open cover of $\mathbb{R}$ (indeed, $\mathbb{R} = E \cup E^c$, and for all $x \in E$, $x \in B(x, \delta_x)$, and for all $b_n \in E^c$, $b_n \in B(b_n, \delta_{b_n})$), so in particular, it forms an open cover of $[a, b]$. Since $[a, b]$ is compact, we can extract a finite subcover, thus there exists a finite set of points $a \le x_1 \le \cdots \le x_K \le b$ such that $[a, b] \subseteq \cup_{k=1}^{K} B(x_k, \delta_{x_k})$. Denote by $B_k = B(x_k, \delta_{x_k})$.

We assume, without loss of generality, that the set $\{B_1, \ldots, B_K\}$ is minimal (for inclusion). In this case, we have for all k, $B_k \cap B_{k+1} \neq \emptyset$ [2]. For all $k \in \{1, \ldots, K-1\}$, let $p_k \in B_k \cap B_{k+1}$, and let $p_0 = a$ and $p_K = b$. This defines a partition

$$a = p_0 \le p_1 \le \cdots \le p_K = b$$

such that for all $k \in \{1, \ldots, K\}$, p_{k-1} and p_k are contained in B_k. Therefore we have

$$F(b) - F(a) = \sum_{k=1}^{K} |F(p_k) - F(p_{k-1})|$$

and for all k, $|F(p_k) - F(p_{k-1})| \le |F(p_k) - F(x_k)| + |F(x_k) - F(p_{k-1})|$, two cases can occur

- if x_k is a good point (i.e. $x_k \in E$), then

$$|F(p_k) - F(x_k)| + |F(x_k) - F(p_{k-1})| \le \epsilon |p_k - x_k| + \epsilon |x_k - p_{k-1}| = \epsilon |p_k - p_{k-1}|$$

- if x_k is a bad point, i.e. $x_k = b_n$ for some n, then

$$|F(p_k) - F(x_k)| + |F(x_k) - F(p_{k-1})| \le 2\epsilon / 2^n$$

therefore

$$\begin{aligned} F(b) - F(a) &= \sum_{k=1}^{K} |F(p_k) - F(p_{k-1})| \\ &\le \sum_{k=1}^{K} \epsilon |p_k - p_{k-1}| + \sum_{n \in \mathbb{N}} 2\epsilon / 2^n \\ &\le \epsilon |b - 1| + 4\epsilon \end{aligned}$$

since this is true for arbitrary $\epsilon > 0$, we have $F(b) - F(a) = 0$.

[2] if $B_k \cap B_{k+1} = \emptyset$, then $x_k + \delta_{x_k} \le x_{k+1} - \delta_{x_{k+1}}$, and there exists p such that

$$x_k + \delta_{x_k} \le p \le x_{k+1} - \delta_{x_{k+1}}$$

i.e. p is not in $B_k \cup B_{k+1}$. Since we have a cover, there exists k' such that $p \in B_{k'}$, but then we either have $k' > k + 1$, in which case $B_{k'} \supset B_{k+1}$, or $k' < k$, in which case $B_{k'} \supset B_k$. Both cases contradict minimality of the cover.

(4.7) Construct an example of a nondecreasing function $F : \mathbb{R} \to \mathbb{R}$ such that $F'(x) = 0$ for almost every $x \in \mathbb{R}$, but there exists no interval on which F is constant.

answer Let $\mathcal{C}$ be the Cantor set, and F be the Cantor function constructed in the text. We have F is continuous, and $F' = 0$ for almost every x ($F' = 0$ on $\mathcal{C}^c$). Now consider the function

$$G : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto G(x) = \sum_{n \in \mathbb{N}} F(q_n + x) 2^{-n}$$

Then we have

- for all n, $F_n : x \mapsto F(q_n + x)$ is non-decreasing and has derivative $F_n' = 0$ a.e. And for all x, $\sum_n |F_n(x)| \leq \sum_n 2^{-n} \leq 2$. Therefore G is an absolutely convergent sequence of non-decreasing functions, thus by Theorem 7.3 (Fubini's theorem on the differentiation of series of monotone functions) G is differentiable a.e. and for almost every x,

$$G(x) = \sum_n F_n'(x) = 0$$

- G is not constant on any interval. Indeed, we recall that by definition,

$$x > 0 \Rightarrow F(x) > 0$$
$$x \leq \Rightarrow F(x) = 0$$

now let $x < y \in \mathbb{R}$. Then $-y < -x$, and density of $\mathbb{Q}$, there exists $q_m \in \mathbb{Q}$ such that $-y < q_m < -x$. Therefore we have

$$x + q_m < 0$$
$$y + q_m > 0$$

therefore $F(x + q_m) = 0$ and $F(y + q_m) > 0$. Therefore $F_m(y) > F_m(x)$. And since for all $n \neq m$, $F_n(y) \geq F_n(x)$ (each F_n is non decreasing), we have

$$\sum_n F_n(y) > \sum_n F_n(x)$$

therefore $G(y) > G(x)$, which completes the proof

(4.8) Show that the product of two functions of bounded variation is of bounded variation.

proof Notation: if $f : [a,b] \to \mathbb{R}$, then the total variation of f on $[a,b]$ is denoted by $V_{[a,b]}(f) = \sup_{P=(x_0,\ldots,x_n)} \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$ where the supremum is taken over the set of partitions of the segment $[a,b]$.

We first show that a function of bounded variations is bounded. If $f : [a,b] \to \mathbb{R}$ is unbounded, so is the function $x \mapsto |f(x) - f(x_0)|$ for some fixed x_0. Therefore for any $M > 0$, there exists $x \in I$ such $|f(x) - f(x_0)| > M$, therefore the total variation of f is unbounded.

Let f, g be two functions of bounded variations on the interval $[a,b]$.

- since f, g are both bounded, there exist $M, N > 0$ such that $\sup_{x\in[a,b]} |f(x)| \le M$ and $\sup_{x\in[a,b]} |g(x)| \le N$
- and since f and g are of bounded variations, there exist $A, B > 0$ such that $V_{[a,b]}(f) \le A$ and $V_{[a,b]}(g) \le B$.

Let $P = (x_0, \ldots, x_n)$ be a partition of the interval (with $a = x_0 \le x_1 \le \cdots \le x_n = b$). Then we have

$$\begin{aligned}
\sum_{i=1}^n |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| &= \sum_{i=1}^n |f(x_i)(g(x_i) - g(x_{i-1})) + g(x_{i-1})(f(x_i) - f(x_{i-1}))| \\
&\le \sum_{i=1}^n |f(x_i)||g(x_i) - g(x_{i-1})| + |g(x_{i-1})||f(x_i) - f(x_{i-1})| \quad \text{by the triangle inequality} \\
&\le M \sum_{i=1}^n |(g(x_i) - g(x_{i-1}))| + N \sum_{i=1}^n |f(x_i) - f(x_{i-1}))| \\
&\le MB + NA
\end{aligned}$$

therefore the total variation of fg is bounded, which proves the result.

(4.9) Define $f, g : [0,1] \to \mathbb{R}$ by

$$f(x) = x\sin(1/x)$$
$$g(x) = x^2\sin(1/x)$$

for all $x \neq 0$, and $f(0) = g(0) = 0$. Determine which of these functions is of bounded variation on $[0,1]$. Justify.

answer We show that f is not of bounded variations. Let, for integers $k \geq 0$

$$x_k = \frac{1}{\pi/2 + k\pi}$$

we have $1/x_k \geq \pi/2$ thus $x_k \leq 2/\pi < 1$. Thus for all k, $x_k \in (0,1)$. And we have

$$\sin(1/x_k) = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases}$$

therefore for all k, $|f(x_k) - f(x_{k-1})| = |x_k \sin(1/x)k) - x_{k-1}\sin(1/x_{k-1})| = x_k + x_{k-1}$. Now for a given K, consider the partition $P_K = \{0, x_K, \ldots, x_0, 1\}$. Then the total variation of f is greater than

$$\begin{aligned}\sum_{k=1}^{K} |f(x_k) - f(x_{k-1})| &= \sum_{k=1}^{K} x_k + x_{k-1} \\ &\geq \sum_{k=1}^{K} x_k \\ &= \sum_{k=1}^{K} \frac{1}{\pi/2 + k\pi} \\ &\geq \sum_{k=1}^{K} \frac{1}{(k+1)\pi}\end{aligned}$$

which tends to ∞ as $K \to \infty$. Therefore f is not of bounded variation.

We show that g is differentiable on $[0,1]$ and has bounded derivative.

- g is differentiable on $(0,1]$ and for all $x \in (0,1]$,

$$\begin{aligned}g'(x) &= 2x\sin(1/x) + x^2\cos(1/x)(-1/x^2) \\ &= 2x\sin(1/x) - \cos(1/x)\end{aligned}$$

  thus

$$\begin{aligned}|g'(x)| &\leq 2|x\sin(1/x)| + |\cos(1/x)| \\ &\leq 2|x| + 1 \qquad \text{since } |\sin(x)| \leq 1 \text{ and } |\cos(1/x) \leq 1| \\ &\leq 3\end{aligned}$$

- for $x = 0$, we have

$$\lim_{x\to 0} \frac{g(x) - g(0)}{x} = \lim_{x \to 0} x\sin(1/x) = 0$$

  thus g is differentiable at 0 and $g'(0) = 0$.

Thus g is differentiable on $[0,1]$ and g' is bounded. Therefore g is of bounded variations, since for any partition $P = \{x_0, \dots, x_n\}$ with $0 = x_0 \leq \cdots \leq x_n = 1$, we have

$$\begin{aligned}
\sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| &= \sum_{i=1}^{n} |\int_{[x_i, x_{i-1}]} g'(t)dt| \\
&\leq \sum_{i=1}^{n} \int_{[x_i, x_{i-1}]} |g'(t)|dt \\
&= \int_{[0,1]} |g'(t)|dt \\
&\leq M
\end{aligned}$$

where M is an upper bound on $|g'|$.