MATH 202B - Problem Set 2

Walid Krichene (23265217)

February 6, 2013

(2.1) Let (a_n) be a sequence of non-negative numbers satisfying $\sum_n a_n = \infty$. Construct a sequence (f_n) of measurable functions such that $\|f_n\|_1 = a_n$ for all n, and for all x, $(f_n(x))_n$ does not converge.

answer We consider the following:

1. if (a_n) does not converge, then consider a measure space with a single point, say $\{0\}$, with the counting measure, and define $f_n = a_n 1_{\{0\}}$.
2. if (a_n) converges to a positive constant, then consider a measure space with two points, say $\{0, 1\}$, with the counting measure, and define

$$f_n = \begin{cases} a_n 1_{\{0\}} & \text{if } n \text{ is odd} \\ a_n 1_{\{1\}} & \text{if } n \text{ is even} \end{cases}$$

Now consider the case (a_n) converges to 0, and assume $a_n \leq 1 \forall n$ (otherwise since $a_n \to 0$, there exists N such that $\forall n \geq N$ $a_n \leq 1$, and it suffices to do the following analysis for the truncated sequence $(a_n)_{n \geq N}$). Then consider the measure space $([0, 1], \mathcal{B}, \mu)$ where $\mathcal{B}$ is the set of Borel measurable sets, and μ is the Lebesgue measure. Let (b_n) be the sequence of reals in $[0, 1]$ defined by

$$b_n = \sum_{m=1}^{n} a_m - \left\lfloor \sum_{m=1}^{n} a_m \right\rfloor$$

then define the sequence of measurable sets (E_n) by

$$E_n = \begin{cases} [b_n, b_{n+1}) & \text{if } b_n < b_{n+1} \\ [b_n, 1] \cup [0, b_{n+1}) & \text{if } b_n \geq b_{n+1} \end{cases}$$

we have by construction $\mu(E_n) = a_n$. An important observation is that a subset of consecutive intervals $E_n, E_{n+1}, \ldots, E_{n+k}$ are either pairwise disjoint, or they cover the whole interval $[0, 1]$.

Now for all n, let $f_n = 1_{E_n}$. Then we have

- $\|f_n\|_1 = \int_X |1_{E_n}| d\mu = \mu(E_n) = a_n$.
- for all $x \in [0, 1]$, $(f_n(x))_n$ does not converge. Indeed, if we suppose by contradiction that $(f_n(x))_n$ converges for some x, then since every f_n takes values in $\{0, 1\}$, we have $\lim f_n(x)$ is either 0 or 1. We show that both cases lead to a contradiction
    1. if $\lim_n f_n(x) = 0$, then there exists N such that $\forall n \geq N$, $f_n(x) = 0$, i.e. $\forall n \geq N$, $x \notin E_n$. Therefore $x \notin \cup_{n \geq N} E_n$, thus for every $M \geq N$, the intervals $(E_n)_{N \leq n \leq M}$ do not cover the whole interval $[0, 1]$ (their union does not contain x), and by the previous observation, they must be pairwise disjoint. Therefore

$$\sum_{n=N}^{M} a_n = \sum_{n=N}^{M} \mu(E_n) = \mu(\cup_{n=N}^{M} E_n) \leq \mu([0, 1]) = 1$$

for every $M \geq N$. This contradicts the fact that $\sum_n a_n = \infty$

2. if $\lim_n f_n(x) = 1$, then there exists N such that $\forall n \geq N$, $f_n(x) = 1$, i.e. $\forall n \geq N$, $x \in E_n$. Therefore for all $n \geq N$, the two consecutive intervals (E_n, E_{n+1}) are not pairwise disjoint (they both contain x), and by the previous observation, they must therefore cover the entire interval $[0, 1]$. In particular, we must have

$$a_n + a_{n+1} = \mu(E_n) + \mu(E_{n+1}) \geq \mu(E_n \cup E_{n+1}) = 1$$

for all $n \geq N$. This contradicts the fact that $\lim_n a_n = 0$.

(2.2.a) Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $0 < p \leq q \leq \infty$. Then

$$L^q \subset L^p$$

proof Case where $q < \infty$. Let $f \in L^q$. Let $E = \{x \in X : |f(x)| \leq 1\} = f^{-1}([-1, 1])$, E is measurable since f is measurable. Then we have

$$\begin{aligned}
\int_X |f|^p d\mu &= \int_E |f|^p d\mu + \int_{E^c} |f|^p d\mu \\
&\leq \int_E 1 d\mu + \int_{E^c} |f|^{q \cdot p/q} d\mu \\
&\leq \mu(E) + \int_{E^c} |f|^{q \cdot p/q} d\mu
\end{aligned}$$

Then we have for all $x \in E^c$, $|f(x)| > 1$, thus $|f(x)|^q > 1$, and since $p/q \leq 1$, we have $y^{p/q} \leq y$ for $y \geq 1$. Thus

$$|f(x)|^{q \cdot p/q} \leq |f(x)|^q \ \forall x \in E^c$$

therefore

$$\begin{aligned}
\int_X |f|^p d\mu &\leq \mu(E) + \int_{E^c} |f|^q d\mu \\
&\leq \mu(E) + \int_X |f|^q d\mu \\
&\leq \mu(E) + \|f\|_q^q < \infty
\end{aligned}$$

thus $f \in L^p$ and $L^q \subset L^p$.

Case where $q = \infty$ and $0 < p < \infty$. Let $f \in L^\infty$. Then there exists $M > 0$ such that $E = \{x \in X : |f(x)| > M\}$ is a null set (i.e. $|f|$ is bounded by M almost everywhere). Then we have

$$\begin{aligned}
\int_X |f|^p d\mu &= \int_{E^c} |f|^p d\mu && \text{since } E \text{ is a null-set} \\
&\leq \int_{E^c} M^p d\mu \\
&= M^p \mu(E^c) < \infty
\end{aligned}$$

therefore $f \in L^p$ and $L^\infty \subset L^p$.

(2.2.b) Suppose that $\exists \delta > 0$ such that $\forall E \in \mathcal{A},\ \mu(E) > 0 \Rightarrow \mu(E) \geq \delta$ (for example, a counting measure has this property).

- Let $0 < p < \infty$. Show that $L^p \subset L^\infty$.
- Corollary: let $0 < p \leq q \leq \infty$. Show that $L^p(\mu) \subset L^q(\mu)$.

proof

- Let $0 < p < \infty$, and let $f \notin L^\infty$. Then we have for all $M > 0$, $E_M = \{x \in X : |f(x)| > M\}$ is not a null set. Then from the assumption, we have

  $$\forall M > 0,\ \mu(E_M) \geq \delta$$

  and we have for all $M > 0$

  $$\begin{aligned}\int_X |f|^p d\mu &\geq \int_{E_M} |f|^p d\mu \\ &\geq \int_{E_M} M^p d\mu \\ &= M^p \mu(E_M) \\ &\geq M^p \delta\end{aligned}$$

  since this holds for all $M > 0$, we have $\int_X |f|^p d\mu = \infty$, therefore $f \notin L^p$. This shows that $(L^\infty)^c \subset (L^p)^c$. Taking complements gives the desired inclusion.

- Let $0 < p \leq q \leq \infty$, and let $f \in L^p$. By the previous fact, we have $f \in L^\infty$. Thus there exists $M > 0$ such that $\mu(E_M) = 0$, where $E_M = \{x \in X : |f(x)| > M\}$. Thus we have

  $$\begin{aligned}\int_X |f|^q d\mu &= \int_{E_M^c} |f|^q d\mu && \text{since } \mu(E_M) = 0 \\ &= \int_{E_M^c} (|f|^p)^{q/p} d\mu\end{aligned}$$

  For all $x \in E_M^c$, $|f(x)| < M$, thus $\frac{1}{M^p}|f(x)|^p < 1$. And since $y^{q/p} \leq y$ for $y \in [0,1]$ $(q/p \geq 1)$, we have

  $$(\frac{1}{M^p}|f(x)|^p)^{q/p} \leq \frac{1}{M^p}|f(x)|^p\ \forall x \in E_M^c$$

  therefore

  $$\begin{aligned}\int_X |f|^q d\mu &\leq M^{q-p} \int_{E_M^c} |f|^p d\mu \\ &\leq M^{q-p} \|f\|_p^p < \infty\end{aligned}$$

  which proves that $f \in L^q$

(2.3) Let $I \subset \mathbb{R}$ be a closed interval, and $\phi : I \to \mathbb{R}$ be a function twice continuously differentiable on the interior of I, which extends continuously on I. Suppose that $\forall x \in \text{int}(I)$, $\phi''(x) \geq 0$. Then $\forall t \in [0,1]$ and $\forall x, y \in I$

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

as a special case, conclude that $\forall t \in [0,1]$ and $\forall x, y \in I$

$$e^{(tx+(1-t)y} \leq te^x + (1-t)e^y$$

and therefore for all $a, b \geq 0$ and conjugate $p, q \in (0, \infty)$

$$ab \leq p^{-1}a^p + q^{-1}b^q$$

proof Let Δ be the difference between the two terms we seek to compare

$$\begin{aligned} \Delta : [0,1] &\to \mathbb{R} \\ t &\mapsto \phi(t(x-y)+y) - t(\phi(x) - \phi(y)) + \phi(y) \end{aligned}$$

We study variations of Δ. We have Δ is twice continuously differentiable, and for all $t \in [0,1]$

$$\begin{aligned} \Delta'(t) &= (x-y)\phi'(t(x-y)+y) - (\phi(x) - \phi(y)) \\ \Delta''(t) &= (x-y)^2\phi''(t(x-y)+y) \end{aligned}$$

from the assumption $\phi'' \geq 0$, we have $\Delta''(t) \geq 0$, therefore $t \mapsto \Delta'(t)$ is increasing on $[0,1]$.

Next, we have for all $u \in [x,y]$, $\phi'(x) \leq \phi'(u) \leq \phi'(y)$ (ϕ' is increasing). Therefore, integrating over $u \in [x,y]$, we have

$$(y-x)\phi'(x) \leq \phi(y) - \phi(x) \leq (y-x)\phi'(y)$$

thus

$$\begin{aligned} \Delta'(0) &= (x-y)\phi'(y) - (\phi(x) - \phi(y)) \leq 0 \\ \Delta'(1) &= (x-y)\phi'(x) - (\phi(x) - \phi(y)) \geq 0 \end{aligned}$$

Combining these results, we have the following variations of Δ

t	0				1
$\Delta''(t)$			+		
$\Delta'(t)$	$\Delta'(0)$	$-$	0	+	$\Delta'(1)$
$\Delta(t)$	0	↘		↗	0

Δ is non-increasing then non-decreasing on $[0,1]$, with $\Delta(0) = \Delta(1) = 0$, therefore $\Delta(t) \leq 0$ for all $t \in [0,1]$. This proves the desired inequality.

As a special case, let

$$\begin{aligned} \phi : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto e^x \end{aligned}$$

then ϕ satisfies the previous assumptions, with $I = \mathbb{R}$. Then applying the previous inequality to ϕ yields

$$\forall t \in [0,1],\ e^{(tx+(1-t)y} \leq te^x + (1-t)e^y$$

Let $p, q \in (0, \infty)$ be conjugate. Let $a, b > 0$ (the case where $a = 0$ or $b = 0$ is trivial). Then take $x = p \ln a$ and $y = q \ln b$. Applying the previous inequality with $t = 1/p$ and using the fact that $1 - t = 1/q$, we have

$$e^{p^{-1} p \ln a + q^{-1} q \ln b} \leq p^{-1} e^{p \ln a} + q^{-1} e^{q \ln b}$$

i.e.

$$ab \leq p^{-1} a^p + q^{-1} b^q$$

(2.4) Let $p \in (0, 1)$. Show that for any $a, b \geq 0$, $(a + b)^p \leq a^p + b^p$. Conclude that for any $f, g \in L^p$, $\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$.

proof Consider the function

$$\begin{aligned} \psi : [0,1] &\to \mathbb{R} \\ t &\mapsto (a + tb)^p - a^p - (tb)^p \end{aligned}$$

We seek to show that $\psi(1) \leq 0$, and we have $\psi(0) = 0$, therefore it suffices to show that ψ is non-increasing on $[0, 1]$. We have

$$\psi'(t) = pb(a + tb)^{p-1} - pb(tb)^{p-1}$$

but we have $\forall t \in [0, 1]$, $a + tb \geq tb \geq 0$, thus since $p - 1 < 0$, $(a + tb)^{p-1} \leq (tb)^{p-1}$ and it follows that $\psi'(t) \leq 0$, therefore ψ is non-increasing on $[0, 1]$ which concludes the proof of the first part.

Let $f, g \in L^p$. Using the previous inequality, we have for all $x \in X$,

$$|f(x) + g(x)|^p \leq |f(x)|^p + |g(x)|^p$$

integrating, we obtain

$$\int_X |f(x) + g(x)|^p d\mu(x) \leq \int_X |f(x)|^p d\mu(x) + \int_X |g(x)|^p d\mu(x)$$

i.e. $\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$

(2.5) Let $p \in (0,\infty)$. Let (f_n) be a sequence of measurable functions satisfying $\lim_{m,n\to\infty} \|f_m - f_n\|_p = 0$. Prove that there exists a subsequence $(f_{n_k})_k$ that converges a.e., that its limit f belongs to L^p, and that $\|f_n - f\|_p \to 0$.

proof First, we list the following inequalities which hold for $p \in (0,1)$ and for measurable functions f, g

$$\|f+g\|_p^p \leq \|f\|_p^p + \|g\|_p^p \tag{1}$$
$$\big|\|f\|_p^p - \|g\|_p^p\big| \leq \|f-g\|_p^p \tag{2}$$

the first was shown in problem (2.4), and the second follows by applying the first to $(f-g)+g$ and $(g-f)+f$, obtainning

$$\|f\|_p^p \leq \|f-g\|_p^p + \|g\|_p^p$$
$$\|g\|_p^p \leq \|f-g\|_p^p + \|f\|_p^p$$

combining both inequalities gives the result.

Existence of a subsequence that converges a.e.: Construct, by induction, an increasing sequence of indices $(n_k)_k$ such that for each $k \geq 1$,

$$\|f_{n_k} - f_m\|_p \leq 2^{-k} \ \forall m \geq n_k$$

this is possible since $\lim_{n,m} \|f_n - f_m\|_p = 0$.

For all $k \geq 1$, let $g_k = f_{n_k}$, and let $E_k = \{x \in X : |g_{k+1}(x) - g_k(x)| > 2^{-k/2}\}$. Then we have by Chebyshev's inequality

$$\mu(E_k) \leq (2^{k/2})^p \|g_{k+1} - g_k\|_p^p \leq 2^{pk/2} 2^{-pk} = 2^{-pk/2}$$

therefore $\sum_{k=1}^{\infty} \mu(E_k) \leq \sum_{k=1}^{\infty} (2^{-p/2})^k$ which is finite since $2^{-p/2} < 1$. Let

$$B = \{x \in X : x \in E_k \text{ for infinitely many } k\}$$

By Borel-Cantelli's theorem, we have B is a null set. We have for all $x \in B^c$, $\{k : x \in E_k\}$ is finite, i.e. there exists $K \geq 1$ such that $\forall k \geq K$, $x \notin E_k$, and we have for all $j \geq i \geq K$

$$\begin{aligned}|g_i(x) - g_j(x)| &\leq \sum_{k=i}^{j-1} |g_{k+1}(x) - g_k(x)| \\ &\leq \sum_{k=i}^{j-1} (1/2)^k \\ &\leq 2(1/2)^i\end{aligned}$$

which converges to 0 as i tends to infinity, i.e. $(g_k(x))_k$ is a Cauchy sequence (in $\mathbb{R}$) thus it converges. Therefore (g_k) converges almost everywhere (for all $x \in B^c$).

The limit is in L^p: Define f by

$$\begin{aligned} f &: X \to \mathbb{R} \\ x \mapsto f(x) &= \begin{cases} \lim_k g_k(x) & \text{if } x \in B^c \\ 0 & \text{is } x \in B \end{cases}\end{aligned}$$

f is measurable as the limit of measurable functions $\tilde{g} = g.1_{B^c}$. We have by Fatou's lemma

$$\begin{aligned}
\int_X |f|^p d\mu &= \int_{B^c} |f|^p \\
&= \int_{B^c} \liminf_k |g_k| d\mu \\
&\leq \liminf_k \int_{B^c} |g_k| d\mu \\
&\leq \liminf_k \|g_k\|_p^p
\end{aligned}$$

finally, using inequality (2), we have for all $j \geq i$

$$\begin{aligned}
|\|g_j\|_p^p - \|g_i\|_p^p| &\leq \sum_{k=i}^{j-1} |\|g_{k+1}\|_p^p - \|g_k\|_p^p| \\
&\leq \sum_{k=i}^{j-1} \|g_{k+1} - g_k\|_p^p && \text{using (2)} \\
&\leq \sum_{k=i}^{j-1} 2^{-k}
\end{aligned}$$

therefore the sequence of the norms $(\|g_k\|_p^p)_k$ is Cauchy, therefore is is bounded, therefore $\liminf_k \|g_k\|_p^p$ is finite, and so is $\int_X |f|^p d\mu$.

Convergence in "norm" ($\lim_n \|f_n - f\|_p = 0$) First, we observe that it suffices to show that $\lim_k \|g_k - f\|_p = 0$. Then for all $\epsilon > 0$, choose k such that $2^{-k} \leq \epsilon/2$, and $\|g_k - f\|_p \leq \epsilon/2$. It follows that for all $n \geq n_k$

$$\begin{aligned}
\|f_n - f\|_p^p &\leq \|f_n - g_k\|_p^p + \|g_k - f\|_p^p && \text{by (1)} \\
&\leq 2^{-kp} + \epsilon^p/2 \\
&\leq \epsilon^p
\end{aligned}$$

and it follows that $\lim_n \|f_n - f\|_p = 0$.

Let us now show that $\lim_k \|g_k - f\|_p = 0$. In order to use the DCT and exchange limit and integral, we seek to dominate $|f - g_k|^p$. We have for $m \geq k$ and for all $x \in B^c$, $|g_m(x) - g_k(x)| \leq \sum_{i=k}^{m-1} |g_{i+1}(x) - g_i(x)|$. Thus taking the limit as $m \to \infty$ and using continuity, we have for all $x \in B^c$

$$|f(x) - g_k(x)| \leq \sum_{i \geq k} |g_{i+1}(x) - g_i(x)| \leq G(x)$$

where $G(x) = \sum_{i \geq 1} |g_{i+1}(x) - g_i(x)|$. Let G_n be the partial sum $G_n(x) = \sum_{i=1}^{n} |g_{i+1}(x) - g_i(x)|$. Then we have by the MCT

$$\begin{aligned}
\int G^p d\mu &= \int \lim_n G_n^p d\mu && \text{using continuity of } y \mapsto y^p \\
&= \lim_n \int G_n^p d\mu && \text{by the DCT} \\
&= \lim_n \|G_n\|_p^p
\end{aligned}$$

where for all n

$$
\begin{aligned}
\|G_n\|_p^p &\le \sum_{i=1}^{n} \|g_{i+1} - g_i\|_p^p && \text{by applying (1) } n \text{ times} \\
&\le \sum_{i=1}^{n} (2^{-i})^p \\
&\le C
\end{aligned}
$$

where $C = \sum_{i\ge1}(2^{-i})^p < \infty$. Therefore the limit is also bounded by C, and so is $\int G^p d\mu$. This shows that G^p is integrable, and it dominates $|f - g_k|^p$.

Finally, we have

$$
\begin{aligned}
\lim_k \|g_k - f\|_p^p &= \lim_k \int_X |g_k - f|^p d\mu \\
&= \int_X \lim_k |g_k - f|^p d\mu && \text{by the DCT, using the dominator } G^p \\
&= \int_{B^c} \lim_k |g_k - f|^p d\mu && \text{since } B \text{ is a null set} \\
&= 0 && \text{since } (g_k(x))_k \text{ converges to } f(x) \text{ for all } x \in B^c.
\end{aligned}
$$

(2.6)

- (f_n) is said to converge to f in measure if $\forall \epsilon > 0$, $\forall \eta > 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$, $\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \leq \eta$
- (f_n) is said to be Cauchy in measure if $\forall \epsilon > 0$, $\forall \eta > 0$, $\exists N \in \mathbb{N}$ such that for all $m, n \geq N$ $\mu(\{x : |f_n(x) - f_m(x)| > \epsilon\}) \leq \eta$

(2.6.a) Let $p \in [1, \infty]$. Show that if a sequence is Cauchy in L^p, then it is Cauchy in measure.

proof Let (f_n) be a sequence Cauchy in L^p. Fix $\epsilon > 0$, and let $\eta > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $\|f_n - f_m\|_p \leq \eta^{1/p}\epsilon$, then we have by Chebyshev's inequality,

$$\begin{aligned} \mu(\{x \in X : |f_n(x) - f_m(x)| > \epsilon\}) &\leq \epsilon^{-p}\|f_n - f_m\|_p^p \\ &\leq \epsilon^{-p}(\eta^{1/p}\epsilon)^p \\ &= \eta \end{aligned}$$

therefore (f_n) is Cauchy in measure.

(2.6.b) Show that if a sequence is Cauchy in measure, then it has a subsequence which converges a.e.

proof Let (f_n) be a sequence Cauchy in measure. Then construct an increasing sequence of indices $(n_k)_k$, such that for all k,

$$\forall m \geq n_k,\ \mu(\{x : |f_m(x) - f_{n_k}(x)| > 2^{-k}\}) \leq 2^{-k}$$

(simply set both ϵ and η to be 2^{-k} when constructing n_k). Let $g_k = f_{n_k}$, and $E_k = \{x : |g_{k+1}(x) - g_k(x)| > 2^{-k}\}$. By construction, $\mu(E_k) \leq 2^{-k}$, thus $\sum_{k\geq 1}\mu(E_k) < \infty$, and by Borel-Cantelli, the set $B = \{x : x \in E_k \text{ for infinitely many } k\}$ is a null set. Now for all $x \in B^c$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $x \notin E_k$, and we have $\forall n \geq m \geq K$

$$\begin{aligned} |g_n(x) - g_m(x)| &\leq \sum_{k=n}^{m-1} |g_{k+1}(x) - g_k(x)| \\ &\leq \sum_{k=n}^{m-1} 2^{-k} && \text{since } x \notin E_k \\ &< 2.2^{-n} \end{aligned}$$

which converges to 0 as n goes to infinity, therefore $(g_n(x))_n$ is Cauchy in $\mathbb{R}$ and it converges. Since this holds for every $x \in B^c$, $(g_n(x))$ converges for a.e. x.

(2.6.c) By Egorov's theorem, if (f_n) converges to f a.e., then (f_n) converges to f almost uniformly, in the sense that

$$\forall \epsilon > 0,\ \exists E \in \mathcal{A} :\ \mu(E) < \epsilon \text{ and } f_n \to f \text{ uniformly on } E^c$$

show that almost uniform convergence implies converges in measure.

proof Let (f_n) a sequence that converges to f almost uniformly. Let $\epsilon > 0$, and $\eta > 0$. Then there exists measurable $E \subset X$ such that $\mu(E) \leq \eta$, and f_n converges uniformly to f on E^c, in particular, there exists N such that for all $n \geq N$, for all $x \in E^c$, $|f_n(x) - f(x)| \leq \epsilon$, thus for all $n \geq N$

$$\{x : |f_n(x) - f(x)| > \epsilon\} \subseteq E$$

and taking measures, we have

$$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \leq \mu(E) \leq \eta$$