(11.1) Let $X, Y$ be Banach spaces, and let $L \in \mathcal{B}(X,Y)$ be injective. Suppose that for each $n \in \mathbb{N}$, there exists $x_n \in X$ such that $\|x_n\|_X = 1$, and $\|L(x_n)\|_Y < 1/n$. What can one conclude about $L$?

**answer**  $L$ is not surjective. Suppose by contradiction that $L$ is surjective, then $L$ is bijective, and by the open mapping theorem, $L$ is open, and it follows that $L^{-1}$ is continuous (the inverse image of any open ball $B(x,r)$ by $L^{-1}$ is $L(B(x,r))$ and is open), thus is bounded. Therefore there exists $M > 0$ such that for all $y \in Y$, $\|L^{-1}(y)\|_X \leq M\|y\|_Y$, in particular, $1 = \|x_n\| = \|L^{-1}(L(x_n))\|_X \leq M\|L(x_n)\|_Y < M/n$ for all $n$, which leads to a contradiction.

(11.2) Let $X$ be a Banach space and $S \subset X$. Suppose that each element of $X^*$ is bounded on $S$ (i.e. for all $\ell \in X^*$, $\ell(S)$ is a bounded subset of $\mathbb{F}$). Prove that $S$ is a bounded subset of $X$.

**proof**  Regard $S$ as a subset of $X^{**}$, i.e. for all $x \in S$, let $i_x \in X^{**}$ defined by

$$i_x : X^* \to \mathbb{F}$$

$$\ell \mapsto i_x(\ell) = \ell(x)$$

for all $x$, $i_x$ is a bounded linear functional since for all $\ell$,

$$|i_x(\ell)| = |\ell(x)| \leq \|x\|_X \|\ell\|_{X^*}$$

thus

$$\|i_x\|_{X^{**}} = \sup_{\ell \in X^*, \ell \neq 0} \frac{|i_x(\ell)| = |\ell(x)|}{\|\ell\|_{X^*}} \leq \|x\|_X$$

in fact we have equality.

We have $\{i_s, s \in S\}$ is a family of linear operators bounded on $X^*$, since for all $\ell$, $\sup_{s \in S} i_s(\ell) = \sup_{s \in S} \ell(s)$ which is finite by assumption. Therefore by the Uniform Boundedness Principle ($X^*$ is a Banach space), $\{i_s, s \in S\}$ are uniformly bounded, i.e. there exists $M$ such that for all $s \in S$, $\|i_s\|_{X^{**}} \leq M$, i.e. $\|s\|_X \leq M$.

(11.3) Let $X, Y$ be Banach spaces. Let $L \in \mathcal{B}(X,Y)$ be surjective. Let $(y_n)$ be a convergent sequence in $Y$. Show that there exists $M < \infty$ and a convergent sequence $(x_n)$ in $X$ such that $L(x_n) = y_n$ for all $n$.

**proof**  Let $y$ be the limit of $(y_n)$, and let $x \in X$ such that $L(x) = y$ ($L$ is surjective).

Since $L$ is surjective, by the open mapping theorem, $L(B_X(0,1))$ is an open subset of $Y$, and since it contains $0 = L(0)$, there exists $r > 0$ such that $B_Y(0,r) \subset L(B_X(0,1))$
For all $n$ consider $\tilde{y}_n$ defined by

$$\tilde{y}_n = \begin{cases} \frac{r}{2\|y_n - y\|}(y_n - y) & \text{if } y_n \neq y \\ 0 & \text{if } y_n = y \end{cases}$$

We have $\tilde{y}_n \in B_Y(0, r)$, thus there exists $\tilde{x}_n \in B_X(0, 1)$ such that $L(\tilde{x}_n) = \tilde{y}_n$. Let

$$x_n = x + \frac{2\|y_n - y\|}{r} \tilde{x}_n$$

then we have

$$L(x_n) = L(x) + \frac{2\|y_n - y\|}{r} L(\tilde{x}_n) = y + (y_n - y) = y_n$$

And we have for all $n$

$$\|x_n - x\| = \frac{2\|y_n - y\|}{r} \|\tilde{x}_n\| \leq \frac{2\|y_n - y\|}{r}$$

since $\tilde{x}_n \in B(0, 1)$, therefore $\|x_n - x\| \to 0$, which proves that $(x_n)$ converges to $x$.

(11.4) Give an example of a nonlinear function $f : \mathbb{R} \to \mathbb{R}$ whose graph is closed but which is not continuous.

**answer** consider

$$f : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1/x & \text{otherwise} \end{cases}$$

the graph of $f$ is closed: let $(x_n, f(x_n))$ be a sequence of elements in the graph, that converges to $(x, y)$ in $\mathbb{R}^2$. Then $(f(x_n))$ converges to $y$, and is in particular bounded. Thus there exists $M > 0$ such that $f(x_n) \leq M$ for all $n$, i.e. for all $n$, either $x_n \leq 0$ or $x_n \geq 1/M$. Therefore $(x_n)$ has a subsequence $(x_{n_k})$ in $(-\infty, 0]$ or in $[1/M, \infty)$. Since $f$ is continuous on both these intervals, it follows that

$$y = \lim_k f(x_{n_k}) = f(\lim_k x_{n_k}) = f(x)$$

(11.5) Let $X$ be a Banach space and $Y$ a closed subspace. Show that the quotient space $X/Y$ is a Banach space.

**proof** Consider the function

$$N : X/Y \to \mathbb{R}_+$$

$$[x] \mapsto N([x]) = \inf_{y \in Y} \|x + y\|$$

This function is well defined since if $x, x' \in [x]$, then $x - x' \in Y$, and $y \mapsto y + x' - x$ is a bijection from $Y$ to $Y$, thus

$$\inf_{y \in Y} \|x - y\| = \inf_{y \in Y} \|x' - (x' - x) + y\| = \inf_{y' \in Y} \|x' - y'\|$$

$N$ defines a norm since

- if $N([x]) = 0$, then $\inf_{y \in Y} \|x + y\| = 0$, and since $Y$ is closed and $y \mapsto \|x + y\|$ is continuous, the inf is attained, and there exists $y \in Y$ such that $\|x + y\| = 0$. Therefore $y = -x$, and it follows that $-x \in Y$, i.e. $[x] = [0]$. 

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• for all $\alpha \in \mathbb{F}$ with $\alpha \neq 0$, we have $N(\alpha[x]) = N([\alpha x]) = \inf_{y \in Y} \|\alpha x + y\| = |\alpha| \inf_{y \in Y} \|x + \frac{1}{|\alpha|} y\| = |\alpha| N(x)$. This also holds for $\alpha = 0$.

• triangle inequality: let $x, x' \in X$. By definition of $N$, we have for all $y, y' \in Y$
\[
N([x + x']) \leq \|x + x' + y'\| = \|x + x' + y + (y' - y)\| \leq \|x + y\| + \|x' + (y' - y)\|
\]
thus taking the infimum over $y$, we have for all $y'$
\[
N([x + x']) \leq N([x]) + \inf_{y \in Y} \|x + y + (y' - y)\| = N([x]) + \|x' + (y' - y_0)\|
\]
for some $y_0 \in Y$. Finally, taking the infimum over $y'$, we obtain the triangle inequality
\[
N([x + x']) \leq N([x]) + N([x'])
\]

Now consider a Cauchy sequence $([a_n])_n$ in $X/Y$, and let $([b_n])$ be a subsequence of $([a_n])$ such that $N([b_n] - [b_{n+1}]) \leq 2^{-n}$. We define by induction sequence $(y_n)$ in $Y$ and $(x_n)$ in $X$ such that
\[
x_n = b_n - y_n
\]
\[
\|x_n - x_{n+1}\|_X \leq 2^{-n}
\]
• let $x_1 = b_1$ and $y_1 = 0$
• assume $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ constructed, and the above properties verified up to $n$. Then we have
\[
N([b_{n+1}] - [b_n]) = N([b_{n+1}] - [x_n + y_n]) \leq 2^{-n}
\]
thus there exists $y \in Y$ such that $\|b_{n+1} - (x_n + y_n) + y\| \leq 2^{-n}$, thus
\[
\|b_n - x_n - y\| \leq 2^{-n}
\]
\[
\|b_{n+1} - y_n - y_{n+1}\| \leq 2^{-n}
\]
\[
\|x_{n+1} - x_n\| \leq 2^{-n}
\]
as required. This completes the inductive construction.

The sequence $(x_n)$ is Cauchy since $\|x_n - x_m\| \leq \sum_{k=n+1}^m 2^{-k} \leq 2^{-n}$. Thus since $X$ is complete, $(x_n)$ converges to some $x \in X$, and we have $([x_n]) = ([b_n])$ converges to $[x]$ since
\[
d([x_n] - [x]) = \inf_{y \in Y} \|x_n - x + y\| \leq \|x_n - x\|
\]
since $0 \in Y$. This proves that $([b_n])$ converges in $X/Y$, and $([a_n])$ is a Cauchy sequence that has a converging subsequence, thus converges.

(11.6) Let $Y$ be a locally compact Hausdorff space. Let $\mathcal{M}$ be the set of all complex Radon measures on $Y$. Regard $\mathcal{M}$ as a normed linear space over $\mathbb{F} = \mathbb{C}$, with the natural vector space structure and norm $\|\mu\|_\mathcal{M} = |\mu|(Y)$. Prove that $\mathcal{M}$ is complete.

**proof** The Riesz Representation Theorem provides an isometric bijection between the space of complex measures $(\mathcal{M}, \|\cdot\|_\mathcal{M})$ and the space of linear functionals on $C_0(Y)$, $(C_0(Y)^*, \|\cdot\|_*)$
\[
\phi : C_0(Y)^* \to \mathcal{M}
\]
\[
\ell \mapsto \phi(\ell) = \mu \text{ such that } \ell(f) = \int_Y f d\mu
\]
where $C_0(Y)^* = \mathcal{B}(C_0(Y), \mathbb{F})$ is complete since $\mathbb{F}$ is complete. Therefore $(\mathcal{M}, \|\cdot\|_\mathcal{M})$ is isometric to a complete space, and is complete.
A Fréchet space is a vector space over $\mathbb{F}$, is

- equipped with countably many semi-norms $\| \cdot \|_i$ such that for each $x \in F$, if $x \neq 0$ then there exists $i$ such that $\| x \|_i > 0$.
- equipped with $\mathcal{T}$, the weak topology for the functions $x \mapsto \| x \|_i$. $\mathcal{T}$ is the weakest topology which contains all balls
  \[ B_i(z, r) = \{ x : \| x - z \|_i < r \} \]
  for all $z \in X$ and $r > 0$.
- is complete in the following sense: if $(x_n)$ is a sequence that is Cauchy for $\| \cdot \|_i$, for all $i$, then $x_n$ converges in $F$.

The dual space $F^*$ is the vector space of all continuous linear functionals $\ell : F \rightarrow \mathbb{F}$. A natural topology on $F^*$ is the weak topology for the functions $i_x, x \in F$, defined by

\[ i_x : F^* \rightarrow \mathbb{R} \]

\[ \ell \mapsto i_x(\ell) = \ell(x) \]

$C^\infty([0, 1]) = \cap_{k \in \mathbb{N}} C^k([0, 1])$ is a Fréchet space. A question is whether there is a Banach space structure which defines the same topology as defined by the Fréchet space. The answer is no.

(11.7.a) Let $F$ be a Fréchet space. Show that for each $\ell \in F^*$, there exists $C < \infty$ and $N \in \mathbb{N}$ such that

\[ \forall x \in F, \ |\ell(x)| \leq C \sum_{i=1}^{N} \| x \|_i \]

**proof** Let $B = \{ B_i(x, r), i \in \mathbb{N}, x \in X, r > 0 \}$. Then a base for the topology $\mathcal{T}$ is finite intersections of elements in $B$.

Since $\ell$ is bounded, it is continuous. Thus $\ell^{-1}(B(0, 1))$ is open, and since it contains $0$, there exists an open neighborhood of $0$ contained in $\ell^{-1}(B(0, 1))$, thus there exists $i_1, \ldots, i_K, x_1, \ldots, x_K \in X$, and $r_1, \ldots, r_K > 0$ such that

\[ \ell^{-1}(B(0, 1)) \supset \cap_{k=1}^{K} B_{i_k}(x_k, r_k) \]

and since $0 \in \ell^{-1}(B(0, 1))$, we have for all $k$, $0 \in B_{i_k}(x_k, r_k)$, thus $\| x_k - 0 \| < r_k$. Let $r_k' = r_k - \| x_k \| > 0$. Then for all $x \in B_k(0, r_k')$, $\| x - x_k \| \leq \| x - 0 \| + \| 0 - x_k \| < r_k' + \| x_k \| = r_k$, i.e.

\[ B_k(0, r_k') \subset B(x_k, r_k) \]

therefore $\ell^{-1}(B(0, 1)) \supset \cap_{k=1}^{K} B_{i_k}(0, r_k') \supset \cap_{k=1}^{K} B_{i_k}(0, \epsilon)$ where $\epsilon = \min_k r_k'$. Claim:

\[ \forall x \in F, \ |\ell(x)| \leq \frac{1}{\epsilon} \sum_{k=1}^{K} \| x \|_{i_k} \]

(this shows the result, since it suffice to take $C = \frac{1}{\epsilon}$ and $N = \max_k i_k$).

Let $x \in X$. If there exists $k \in \{ 1, \ldots, K \}$ such that $\| x \|_{i_k} > 0$, then let

\[ \tilde{x} = \frac{\epsilon x}{\sum_k \| x \|_{i_k}} \]

then we have

\[ \| \tilde{x} \|_{i_k} \leq \epsilon \forall k \in \{ 1, \ldots, K \} \]
i.e. \( \tilde{x} \in \bigcap_{k=1}^{K} B_{ik}(0, \epsilon) \) thus \( |\ell(\tilde{x})| \leq 1 \), and by linearity of \( \ell \)

\[
|\ell(x)| = \frac{1}{\epsilon} \sum_{k=1}^{K} \|x\|_{ik} |\ell(\tilde{x})|
\]

\[
\leq \frac{1}{\epsilon} \sum_{k=1}^{K} \|x\|_{ik}
\]

if \( \|x\|_{ik} = 0 \) for all \( k \in \{1, \ldots, K\} \), then for all \( r > 0 \), \( \|rx\|_{ik} = 0 \), and \( rx \in \bigcap_{k=1}^{K} B_{ik}(0, \epsilon) \), thus

\[
|\ell(x)| = \frac{1}{r} |\ell(rx)| \leq \frac{1}{r}
\]

and since this holds for arbitrary \( r \), we have \( \ell(x) = 0 \), which completes the proof.

(11.7.b) Uniform Boundedness Principle for Fréchet space: Let \( F \) be a Fréchet space, and \( S \subset F^* \) be a family of continuous linear functionals which is pointwise bounded, i.e.

\[
\forall x \in F, \exists A_x \text{ such that } \forall \ell \in S, |\ell(x)| \leq A_x
\]

Then \( S \) is uniformly bounded in this sense

\[
\exists N, \exists C < \infty \text{ such that } \forall \ell \in S, \forall x \in F, |\ell(x)| \leq C \sum_{i=1}^{N} \|x\|_{i}
\]

(hint: show that the topology for \( F \) is that of a metric space, and apply the Baire category theorem)

**proof** For all \( i \), define the pseudo-norm \( N_i(x) = \frac{\|x\|_{i}}{1+\|x\|_{i}} \). Then define

\[
d : F \to \mathbb{R}_+
\]

\[
x \mapsto d(x) = \sum_{n=1}^{\infty} 2^{-n} N_i(x)
\]

(the sum converges since for all \( i \), \( N_i(x) \in [0,1] \)). Then \( d \) is a metric, and the topology of \( F \) is the same as the topology induced by \( d \).

Now \( (F, d) \) is a metric space, and it is in particular complete, and we can apply the Baire Category theorem on the sets

\[
E_n = \{ x \in F : |\ell(x)| \leq n \ \forall \ell \in S \}
\]

\( E_n \) are closed, and \( X = \bigcup_{n \in \mathbb{N}} E_n \). Therefore by the Baire category theorem, there exists \( n \) such that \( E_n \) has non empty interior, therefore the exists \( n \), and there exists \( x_0 \) and an open neighborhood \( U \) of \( x_0 \) such that \( \forall x \in U, |\ell(x)| < n \) for all \( \ell \in S \). Similarly to the argument in (a), the open neighborhood \( U \) contains a finite intersection of balls \( B_{ik}(x_k, r_k) \), and shifting the centers to \( x_0 \) (with the appropriate adjustment of the radii), we have

\[
\bigcap_{\ell \in S} \ell^{-1}(B(0, n)) \ni \bigcap_{k=1}^{K} B_{ik}(x_k, r_k) \supset \bigcap_{k=1}^{K} B_{ik}(x_0, r_k') \supset \bigcap_{k=1}^{K} B_{ik}(x_0, \epsilon)
\]

with \( \epsilon = \min_k r_k' \). The rest of the argument is the same as in (a), and shows that for all \( \ell \in S, \forall x \in F \)

\[
|\ell(x)| \leq \frac{1}{\epsilon} \sum_{k=1}^{K} \|x\|_{ik}
\]