MATH 202B - Problem Set 11

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(11.1) Let X, Y be Banach spaces, and let $L \in \mathcal{B}(X, Y)$ be injective. Suppose that for each $n \in \mathbb{N}$, there exists $x_n \in X$ such that $\|x_n\|_X = 1$, and $\|L(x_n)\| < 1/n$. What can one conclude about L?

answer L is not surjective. Suppose by contradiction that L is surjective, then L is bijective, and by the open mapping theorem, L is open, and it follows that L^{-1} is continuous (the inverse image of any open ball $B(x, r)$ by L^{-1} is $L(B(x, r))$ and is open), thus is bounded. Therefore there exists $M > 0$ such that for all $y \in Y$, $\|L^{-1}(y)\|_X \leq M\|y\|_Y$, in particular,

$$1 = \|x_n\| = \|L^{-1}(L(x_n))\|_X \leq M\|L(x_n)\|_Y < M/n$$

for all n, which leads to a contradiction.

(11.2) Let X be a Banach space and $S \subset X$. Suppose that each element of X^* is bounded on S (i.e. for all $\ell \in X^*$, $\ell(S)$ is a bounded subset of $\mathbb{F}$). Prove that S is a bounded subset of X.

proof Regard S as a subset of X^{**}, i.e. for all $x \in S$, let $i_x \in X^{**}$ defined by

$$\begin{aligned} i_x : X^* &\to \mathbb{F} \\ \ell &\mapsto i_x(\ell) = \ell(x) \end{aligned}$$

for all x, i_x is a bounded linear functional since for all ℓ,

$$|i_x(\ell)| = |\ell(x)| \leq \|x\|_X \|\ell\|_{X^*}$$

thus

$$\|i_x\|_{X^{**}} = \sup_{\ell \in X^*, \ell \neq 0} \frac{|i_x(\ell)| = |\ell(x)|}{\|\ell\|_{X^*}} \leq \|x\|_X$$

in fact we have equality.

We have $\{i_s, s \in S\}$ is a family of linear operators bounded on X^*, since for all ℓ, $\sup_{s \in S} i_x(\ell) = \sup_{s \in S} \ell(s)$ which is finite by assumption. Therefore by the Uniform Boundedness Principle (X^* is a Banach space), $\{i_s, s \in S\}$ are uniformly bounded, i.e. there exists M such that for all $s \in S$, $\|i_s\|_{X^{**}} \leq M$, i.e. $\|s\|_X \leq M$.

(11.3) Let X, Y be Banach spaces. Let $L \in \mathcal{B}(X, Y)$ be surjective. Let (y_n) be a convergent sequence in Y. Show that there exists $M < \infty$ and a convergent sequence (x_n) in X such that $L(x_n) = y_n$ for all n.

proof Let y be the limit of (y_n), and let $x \in X$ such that $L(x) = y$ (L is surjective).

Since L is surjective, by the open mapping theorem, $L(B_X(0, 1))$ is an open subset of Y, and since it contains $0 = L(0)$, there exists $r > 0$ such that $B_Y(0, r) \subset L(B_X(0, 1))$

For all n consider $\tilde{y}_n$ defined by

$$\tilde{y}_n = \begin{cases} \frac{r}{2\|y_n - y\|}(y_n - y) & \text{if } y_n \neq y \\ 0 & \text{if } y_n = y \end{cases}$$

We have $\tilde{y}_n \in B_Y(0, r)$, thus there exists $\tilde{x}_n \in B_X(0, 1)$ such that $L(\tilde{x}_n) = \tilde{y}_n$. Let

$$x_n = x + \frac{2\|y_n - y\|}{r}\tilde{x}_n$$

then we have

$$L(x_n) = L(x) + \frac{2\|y_n - y\|}{r}L(\tilde{x}_n) = y + (y_n - y) = y_n$$

And we have for all n

$$\begin{aligned} \|x_n - x\| &= \|\frac{2\|y_n - y\|}{r}\tilde{x}_n\| \\ &\leq \frac{2\|y_n - y\|}{r} \end{aligned}$$

since $\tilde{x}_n \in B(0, 1)$, therefore $\|x_n - x\| \to 0$, which proves that (x_n) converges to x.

(11.4) Give an example of a nonlinear function $f : \mathbb{R} \to \mathbb{R}$ whose graph is closed but which is not continuous.

answer consider

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1/x & \text{otherwise} \end{cases} \end{aligned}$$

the graph of f is closed: let $(x_n, f(x_n))$ be a sequence of elements in the graph, that converges to (x, y) in $\mathbb{R}^2$. Then $(f(x_n))$ converges to y, and is in particular bounded. Thus there exists $M > 0$ such that $f(x_n) \leq M$ for all n, i.e. for all n, either $x_n \leq 0$ or $x_n \geq 1/M$. Therefore (x_n) has a subsequence (x_{n_k}) in $(-\infty, 0]$ or in $[1/M, \infty)$. Since f is continuous on both these intervals, it follows that

$$y = \lim_k f(x_{n_k}) = f(\lim_k x_{n_k}) = f(x)$$

(11.5) Let X be a Banach space and Y a closed subspace. Show that the quotient space X/Y is a Banach space.

proof Consider the function

$$\begin{aligned} N : X/Y &\to \mathbb{R}_+ \\ [x] &\mapsto N([x]) = \inf_{y \in Y} \|x + y\| \end{aligned}$$

This function is well defined since if $x, x' \in [x]$, then $x - x' \in Y$, and $y \mapsto y + x' - x$ is a bijection from Y to Y, thus

$$\inf_{y \in Y} \|x - y\| = \inf_{y \in Y} \|x' - (x' - x + y)\| = \inf_{y' \in Y} \|x' - y'\|$$

N defines a norm since

- if $N([x]) = 0$, then $\inf_{y \in Y} \|x + y\| = 0$, and since Y is closed and $y \mapsto \|x + y\|$ is continuous, the inf is attained, and there exists $y \in Y$ such that $\|x + y\| = 0$. Therefore $y = -x$, and it follows that $-x \in Y$, i.e. $[x] = [0]$.

- for all $\alpha \in \mathbb{F}$ with $\alpha \neq 0$, we have $N(\alpha[x]) = N([\alpha x]) = \inf_{y \in Y} \|\alpha x + y\| = |\alpha| \inf_{y \in Y} \|x + \frac{1}{|\alpha|} y\| = |\alpha| N([x])$. This also holds for $\alpha = 0$.
- triangle inequality: let $x, x' \in X$. By definition of N, we have for all $y, y' \in Y$

  $$N([x + x']) \leq \|x + x' + y'\| = \|x + x' + y + (y' - y)\| \leq \|x + y\| + \|x' + (y' - y)\|$$

  thus taking the infimum over y, we have for all y'

  $$N([x + x']) \leq N([x]) + \inf_{y \in V} \|x' + (y' - y)\| = N([x]) + \|x' + (y' - y_0)\|$$

  for some $y_0 \in Y$. Finally, taking the infimum over y', we obtain the triangle inequality

  $$N([x + x']) \leq N([x]) + N([x'])$$

Now consider a Cauchy sequence $([a_n])_n$ in X/Y, and let $([b_n])$ be a subsequence of $([a_n])$ such that $N([b_n] - [b_{n+1}]) \leq 2^{-n}$. We define by induction sequence (y_n) in Y and (x_n) in X such that

$$x_n = b_n - y_n$$
$$\|x_n - x_{n+1}\|_X \leq 2^{-n}$$

- let $x_1 = b_1$ and $y_1 = 0$
- assume $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ constructed, and the above properties verified up to n. Then we have

  $$N([b_{n+1}] - [b_n]) = N([b_{n+1}] - [x_n + y_n]) \leq 2^{-n}$$

  thus there exists $y \in Y$ such that $\|b_{n+1} - (x_n + y_n) + y\| \leq 2^{-n}$, thus

  $$\|b_{n+1} - (y_n - y) - x_n\| \leq 2^{-n}$$

  letting $y_{n+1} = y_n - y$, and $x_{n+1} = b_{n+1} - y_{n+1}$, we have

  $$\|x_{n+1} - x_n\| \leq 2^{-n}$$

  as required. This completes the inductive construction.

The sequence (x_n) is Cauchy since $\|x_n - x_m\| \leq \sum_{k=n+1}^{m} 2^{-k} \leq 2^{-n}$. Thus since X is complete, (x_n) converges to some $x \in X$, and we have $([x_n]) = ([b_n])$ converges to $[x]$ since

$$d([x_n] - [x]) = \inf_{y \in Y} \|x_n - x + y\| \leq \|x_n - x\|$$

since $0 \in Y$. This proves that $([b_n])$ converges in X/Y, and $([a_n])$ is a Cauchy sequence that has a converging subsequence, thus converges.

(11.6) Let Y be a locally compact Hausdorff space. Let $\mathcal{M}$ be the set of all complex Radon measures on Y. Regard $\mathcal{M}$ as a normed linear space over $\mathbb{F} = \mathbb{C}$, with the natural vector space structure and norm $\|\mu\|_{\mathcal{M}} = |\mu|(Y)$. Prove that $\mathcal{M}$ is complete.

proof The Riesz Representation Theorem provides an isometric bijection between the space of complex measures $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ and the space of linear functionals on $C_0(Y)$, $(C_0(Y)^*, \|\cdot\|_*)$

$$\phi : C_0(Y)^* \to \mathcal{M}$$
$$\ell \mapsto \phi(\ell) = \mu \text{ such that } \ell(f) = \int_Y f d\mu$$

where $C_0(Y)^* = \mathcal{B}(C_0(Y), \mathbb{F})$ is complete since $\mathbb{F}$ is complete. Therefore $(\mathcal{M}, \|\cdot\|_{\mathcal{M}}$ is isometric to a complete space, and is complete.

(11.7) A seminor on a vector space X over $\mathbb{F} = \mathbb{C}$ is a function $\|\cdot\| : X \to [0,\infty)$ which satisfies the conditions for a norm except definiteness.

A Fréchet space $\mathcal{F}$ is a vector space over $\mathbb{F}$, is

- equipped with countably many semi-norms $\|\cdot\|_i$ such that for each $x \in \mathcal{F}$, if $x \neq 0$ then there exists i such that $\|x\|_i > 0$.
- equipped with $\mathcal{T}$, the weak topology for the functions $x \mapsto \|x\|_i$. $\mathcal{T}$ is the weakest topology which contains all balls
$$B_i(z,r) = \{x : \|x - z\|_i < r\}$$
for all $z \in X$ and $r > 0$.
- is complete in the following sense: if (x_n) is a sequence that is Cauchy for $\|\cdot\|_i$ for all i, then x_n converges in $\mathcal{F}$.

The dual space $\mathcal{F}^*$ is the vector space of all continuous linear functionals $\ell : \mathcal{F} \to \mathbb{F}$. A natural topology on $\mathcal{F}^*$ is the weak topology for the functions $i_x, x \in \mathcal{F}$, defined by

$$\begin{aligned} i_x : \mathcal{F}^* &\to \mathbb{R} \\ \ell &\mapsto i_x(\ell) = \ell(x) \end{aligned}$$

$C^\infty([0,1]) = \cap_{k\in\mathbb{N}} C^k([0,1])$ is a Fréchet space. A question is whether there is a Banach space structure which defines the same topology as defined by the Fréchet space. The answer is no.

(11.7.a) Let $\mathcal{F}$ be a Fréchet space. Show that for each $\ell \in \mathcal{F}^*$, there exists $C < \infty$ and $N \in \mathbb{N}$ such that

$$\forall x \in \mathcal{F},\ |\ell(x)| \le C \sum_{i=1}^{N} \|x\|_i$$

proof Let $B = \{B_i(x,r), i \in \mathbb{N}, x \in X, r > 0\}$. Then a base for the topology $\mathcal{T}$ is finite intersections of elements in B.

Since ℓ is bounded, it is continuous. Thus $\ell^{-1}(B(0,1))$ is open, and since it contains 0, there exists an open neighborhood of 0 contained in $\ell^{-1}(B(0,1))$, thus there exists $i_1,\ldots,i_K$, $x_1,\ldots,x_K \in X$, and $r_1,\ldots,r_K > 0$ such that
$$\ell^{-1}(B(0,1)) \supset \cap_{k=1}^K B_{i_k}(x_k, r_k)$$
and since $0 \in \ell^{-1}(B(0,1))$, we have for all k, $0 \in B_{i_k}(x_k,r_k)$, thus $\|x_k - 0\| < r_k$. Let $r'_k = r_k - \|x_k\| > 0$. Then for all $x \in B_k(0,r'_k)$, $\|x - x_k\| \le \|x - 0\| + \|0 - x_k\| < r'_k + \|x_k\| = r_k$, i.e.

$$B_k(0,r'_k) \subset B(x_k,r_k)$$

therefore $\ell^{-1}(B(0,1)) \supset \cap_{k=1}^K B_{i_k}(0,r'_k) \supset \cap_{k=1}^K B_{i_k}(0,\epsilon)$ where $\epsilon = \min_k r'_k$. Claim:

$$\forall x \in \mathcal{F},\ |\ell(x)| \le \frac{1}{\epsilon} \sum_{k=1}^{K} \|x\|_{i_k}$$

(this shows the result, since it suffice to take $C = \frac{1}{\epsilon}$ and $N = \max_k i_k$).

Let $x \in X$. If there exists $k \in \{1,\ldots,K\}$ such that $\|x\|_{i_k} > 0$, then let

$$\tilde{x} = \frac{\epsilon x}{\sum_k \|x\|_{i_k}}$$

then we have
$$\|\tilde{x}\|_{i_k} \le \epsilon\ \forall k \in \{1,\ldots,K\}$$

i.e. $\tilde{x} \in \cap_{k=1}^K B_{i_k}(0,\epsilon)$ thus $|\ell(\tilde{x})| \leq 1$, and by linearity of ℓ

$$\begin{aligned}|\ell(x)| &= \frac{1}{\epsilon}\sum_{k=1}^K \|x\|_{i_k}|\ell(\tilde{x})| \\ &\leq \frac{1}{\epsilon}\sum_{k=1}^K \|x\|_{i_k}\end{aligned}$$

if $\|x\|_{i_k} = 0$ for all $k \in \{1,\dots,K\}$, then for all $r > 0$, $\|rx\|_{i_k} = 0$, and $rx \in \cap_{k=1}^K B_{i_k}(0,\epsilon)$, thus

$$|\ell(x)| = \frac{1}{r}|\ell(rx)| \leq \frac{1}{r}$$

and since this holds for arbitrary r, we have $\ell(x) = 0$, which completes the proof.

(11.7.b) Uniform Boundedness Principle for Fréchet space: Let $\mathcal{F}$ be a Fréchet space, and $S \subset \mathcal{F}^*$ be a family of continuous linear functionals which is pointwise bounded, i.e.

$$\forall x \in \mathcal{F},\ \exists A_x \text{ such that } \forall \ell \in S,\ |\ell(x)| \leq A_x$$

Then S is uniformly bounded in this sense

$$\exists N, \exists C < \infty \text{ such that } \forall \ell \in S,\ \forall x \in \mathcal{F},\ |\ell(x)| \leq C\sum_{i=1}^N \|x\|_i$$

(hint: show that the topology for $\mathcal{F}$ is that of a metric space, and apply the Baire category theorem)

proof For all i, define the pseudo-norm $N_i(x) = \frac{\|x\|_i}{1+\|x\|_i}$. Then define

$$\begin{aligned}d : \mathcal{F} &\to \mathbb{R}_+ \\ x &\mapsto d(x) = \sum_{n=1}^\infty 2^{-n} N_i(x)\end{aligned}$$

(the sum converges since for all i, $N_i(x) \in [0,1]$). Then d is a metric, and the topology of $\mathcal{F}$ is the same as the topology induced by d.

Now $(\mathcal{F}, d)$ is a metric space, and it is in particular complete, and we can apply the Baire Category theorem on the sets

$$E_n = \{x \in \mathcal{F} : |\ell(x)| \leq n\ \forall \ell \in S\}$$

E_n are closed, and $X = \cup_{n\in\mathbb{N}} E_n$. Therefore by the Baire category theorem, there exists n such that E_n has non empty interior, therefore the exists n, and there exists x_0 and an open neighborhood U of x_0 such that $\forall x \in U$, $|\ell(x)| < n$ for all $\ell \in S$. Similarly to the argument in (a), the open neighborhood U contains a finite intersection of balls $B_{i_k}(x_k, r_k)$, and shifting the centers to x_0 (with the appropriate adjustment of the radii), we have

$$\cap_{\ell\in S}\ell^{-1}(B(0,n)) \ni \cap_{k=1}^K B_{i_k}(x_k, r_k) \supset \cap_{k=1}^K B_{i_k}(x_0, r_k') \supset \cap_{k=1}^K B_{i_k}(x_0, \epsilon)$$

with $\epsilon = \min_k r_k'$. The rest of the argument is the same as in (a), and shows that for all $\ell \in S$, $\forall x \in \mathcal{F}$

$$|\ell(x)| \leq \frac{1}{\epsilon}\sum_{k=1}^K \|x\|_{i_k}$$