MATH 202B - Problem Set 1

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(1.1) Let X be an uncountable set. Let $\mathcal{A} \subset \mathcal{P}(X)$ be the σ-algebra consisting of all sets which are either countable, or have countable complement. Let $\Delta = \{(x, y) \in X \times Y : y = x\}$. Then

(1.1.a) Δ does not belong to the product σ-algebra $\mathcal{A} \times \mathcal{A}$

proof We show that $\forall E \in \mathcal{A} \times \mathcal{A}$, there exists $F \subseteq X$ such that for all but countably many $x \in X$, $E_x = F$.
Let $\mathcal{B}$ be the set

$$\mathcal{B} = \{E \in \mathcal{P}(X \times X) | \exists F \subset X : \forall \text{ but countably many } x \in X, E_x = F\}$$

Then $\mathcal{B}$ contains rectangles: let $A, B \in \mathcal{A}$, and $E = A \times B$. If A is countable, then $E_x = \emptyset$ for all $x \notin A$, and if A^c is countable, then $E_x = B$ for all $x \notin A^c$.

We also have $\mathcal{B}$ is a σ-algebra since it is

- closed under countable union: let (E_n) be a sequence of elements of $\mathcal{B}$, and for all $n \in \mathbb{N}$, let G_n be a countable subset of X and $F_n \subseteq X$ such that $\forall x \notin X_n$, $(E_n)_x = F_n$. Now let $E = \cup_n E_n$, $G = \cup_n G_n$ and $F = \cup_n F_n$. Then we have

  $$\forall x \notin G,\ E_x = (\cup_n E_n)_x = \cup_n (E_n)_x = \cup_n F_n = F$$

  therefore $E \in \mathcal{B}$
- closed under taking the complement: let $E \in \mathcal{B}$ and let $G \subseteq X$ countable and $F \subseteq X$ such that $\forall x \notin G$, $E_x = F$. Then

  $$\forall x \notin G,\ (E^c)_x = (E_x)^c = F^c$$

  therefore $E^c \in \mathcal{B}$.

Therefore the smallest σ-algebra containing rectangles also satisfies the property: $\forall E \in \mathcal{A} \times \mathcal{B}$, $\exists F \subseteq X$ and a countable $G \subseteq X$ such that $\forall x \notin G$, $E_x = F$. This is not the case for Δ.

(1.1.b) Let $x \in X$, and let $E = \{x\} \times X$. Then E is an element of $\mathcal{A} \times \mathcal{A}$ since it is a rectangle (both $\{x\}$ and X are in $\mathcal{A}$) and both E and E^c are uncountable.

(1.2) Let $I = (-1,1)$. Let

$$f : I \times I \to [0,\infty)$$
$$(x,y) \mapsto \frac{1}{y+1} e^{-\frac{|x|}{1+y}}$$

Then $\int_{I\times I} f dm(x,y) < \infty$ and $\int_I f(0,y)dm(y) = \infty$

proof

$$\begin{aligned}
\int_I f(0,y)dm(y) &= \int_I \frac{1}{1+y} dm(y) \\
&= \lim_n \int_{[} -1+1/n, 1] \frac{1}{1+y} dm(y) \\
&= \lim_n \ln(2) - \ln(1/n) \\
&= \infty
\end{aligned}$$

and

$$\begin{aligned}
\int_{I\times I} f(x,y)dm(x,y) &= \int_I \int_I \frac{1}{1+y} e^{-\frac{|x|}{y+1}} dm(x)dm(y) && \text{by Fubini's theorem} \\
&= \int_I 2 \int_{[0,1]} \frac{1}{1+y} e^{-\frac{x}{y+1}} dm(x)dm(y) && \text{by symmetry} \\
&= 2\int_I (1 - e^{-\frac{1}{y+1}})dm(y) \\
&< \infty
\end{aligned}$$

(1.3) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Let λ be a measure defined on $\mathcal{A} \times \mathcal{B}$ such that for every measurable rectangle $A \times B$,

$$\lambda(A \times B) = \mu(A)\nu(B)$$

Then for every $E \in \mathcal{A} \times \mathcal{B}$, we have

$$\lambda(E) = \mu \times \nu(E)$$

proof We first partition the sets X, Y into

$$X = \cup_n X_n \text{ disjointly}$$
$$Y = \cup_n Y_n \text{ disjointly}$$

where for all n, X_n and Y_n have finite measure. Then

$$X \times Y = \cup_{n,m} X_n \times Y_m \text{ disjointly}$$

Let

$$\mathcal{M} = \{M | \lambda(M) = \mu \times \nu(M) \text{ and } \lambda(M \cap (X_n \times Y_m)) = \mu \times \nu(M \times (X_n \times Y_m)) \forall n, m\}$$

and let $\mathcal{C}$ be the algebra of all finite disjoint unions of rectangles. We show that $\mathcal{M}$ contains $\mathcal{C}$. Let $M \in \mathcal{C}$, $M = \cup_{k=1}^K A_k \times B_k$ disjointly. Then

$$\begin{aligned}
\lambda(M) &= \lambda(\cup_k A_k \times B_k) \\
&= \sum_k \lambda(A_k \times B_k) \\
&= \sum_k \mu(A_k)\nu(B_k) \\
&= \sum_k \mu \times \nu(A_k \times B_k) \\
&= \mu \times \nu(\cup_k A_k \times B_k) \\
&= \mu \times \nu(M)
\end{aligned}$$

this also holds for $M \cap (X_n \times Y_m)$, since it is also a finite disjoint union of rectangles. Therefore, $\mathcal{C} \subseteq \mathcal{M}$.

Next, we have $\mathcal{M}$ is a monotone class since

- it is closed under ascending countable union: let $M_1 \subseteq M_2 \subseteq \ldots$ be a nested sequence of elements of $\mathcal{M}$, and let $M = \cup_n M_n$. Then

$$\begin{aligned}
\lambda(M) &= \lim_n \lambda(M_n) \text{by } \sigma\text{-additivity of } \lambda \\
&= \lim_n \mu \times \nu(M_n) \\
&= \mu \times \nu(M) \text{by } \sigma\text{-additivity of } \mu \times \nu
\end{aligned}$$

  and similarly for $M \cap X_n \times Y_m$

- it is closed under descending countable intersection: let $M_1 \supseteq M_2 \supseteq \ldots$ be a nested sequence of elements of $\mathcal{M}$, and let $M = \cap_n M_n$. Then

$$\begin{aligned}
\lambda(M) &= \lambda(\cup_{n,m} M \cap (X_n \times Y_m)) \\
&= \sum_{n,m} \lambda(M \cap (X_n \times Y_m)) && \text{since the union is disjoint} \\
&= \sum_{n,m} \lim_k \lambda(M_k \cap (X_n \times Y_m)) && \text{since } X_n \times Y_m \text{ has finite measure} \\
&= \sum_{n,m} \lim_k \mu \times \nu(M_k \cap (X_n \times Y_m)) \\
&= \sum_{n,m} \mu \times \nu(M \cap (X_n \times Y_m)) \\
&= \mu \times \nu(M)
\end{aligned}$$

  and similarly for $M \cap X_n \times Y_m$

Therefore by the monotone class lemma, the smallest monotone class that contains $\mathcal{C}$ is a σ-algebra, which in turn contains the smallest such σ-algebra, i.e. $\mathcal{A} \times \mathcal{B} \subseteq \mathcal{M}$, which proves the desired result.

(1.4) We have

$$\lim_{T\to\infty}\int_0^T \frac{\sin(x)}{x}dx = \frac{\pi}{2}$$

proof Let $T > 0$. Using the continuous extension of $\frac{sin(x)}{x}$ at 0 and the fact that Riemann and Lebesgue integrals agree for continuous functions on closed intervals,

$$\int_0^T \frac{sin(x)}{x}dx = \int_{[0,T]} \frac{sin(x)}{x}dm(x)$$

then we observe that

$$\begin{aligned}
\frac{1}{x} &= \int_0^\infty e^{-tx}dt \\
&= \lim_N \int_0^N e^{-tx}dt \\
&= \lim_N \int_{[0,N]} e^{-tx}dm(t) \\
&= \lim_N \int_{\mathbb{R}} e^{-tx}1_{[0,N]}(t)dm(t) \\
&= \int_{\mathbb{R}} \lim_N e^{-tx}1_{[0,N]}(t)dm(t) && \text{using the MCT} \\
&= \int_{\mathbb{R}} e^{-tx}dm(t)
\end{aligned}$$

thus

$$\begin{aligned}
\int_0^T \frac{sin(x)}{x}dx &= \int_{[0,T]}\int_{\mathbb{R}} \sin(x)e^{-tx}dm(t)dm(x) \\
&= \int_{\mathbb{R}}\int_{[0,T]} \sin(x)e^{-tx}dm(x)dm(t) && \text{by Fubini's theorem} \\
&= \Im\left(\int_{\mathbb{R}}\int_0^T e^{ix}e^{-tx}dxdm(t)\right)
\end{aligned}$$

Next, we have

$$\begin{aligned}
\int_0^T e^i xe^{-tx}dx &= \left[\frac{1}{i-t}e^{(i-t)x}\right]_0^T \\
&= \frac{1}{i-t}(1-e^{(i-t)T})
\end{aligned}$$

then

$$\begin{aligned}
\lim_{T\to\infty}\int_0^T \frac{\sin(x)}{x}dx &= \lim_T \int_{\mathbb{R}} \Im\left(\frac{1}{i-t}(1-e^{(i-t)T})\right)dm(t) \\
&= \int_{\mathbb{R}} \lim_T \Im\left(\frac{1}{i-t}(1-e^{(i-t)T})\right)dm(t) && \text{using the DCT} \\
&= \int_0^\infty \frac{-1}{1+t^2}dt \\
&= \int_0^{\pi/2} 1du && \text{using the change of variable } t = \tan u \\
&= \pi/2
\end{aligned}$$

(1.5) Let $f : \mathbb{R}^1 \to \mathbb{R}$ and define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x, y) = f(x - y)$$

Then

(1.5.1) if f is Borel measurable, then so is F

proof Let $\mathcal{B}$ denote the set of Borel sets. Let

$$\begin{aligned} g &: \mathbb{R}^2 \to \mathbb{R} \\ &(x, y) \mapsto f(x - y) \end{aligned}$$

Then we have for all $c \in \mathbb{R}$

$$F^{-1}([-\infty, c)) = g^{-1}(f^{-1}([-\infty, c)))$$

where $f^{-1}([-\infty, c)) \in \mathcal{B}$ since f is measurable. Therefore it suffices to show that for all $B \in \mathcal{B}$, $g^{-1}(B) \in \mathcal{B}$. Let

$$\mathcal{C} = \{B \in \mathcal{B} | g^{-1}(B) \in \mathcal{B}\}$$

we have

- $\mathcal{C}$ contains open balls since g is continuous.
- $\mathcal{C}$ is a σ-algebra since it is
  - closed under countable union: if B_n is a sequence of elements of $\mathcal{C}$, then

    $$g^{-1}(\cup_n B_n) = \cup_n g^{-1}(B_n)$$

    which is an element of $\mathcal{B}$ since each term is in $\mathcal{B}$.
  - closed under taking the complement: if $B \in \mathcal{C}$, then $g^{-1}(B^c) = (g^{-1}(B))^c$ which is in $\mathcal{B}$ since $g^{-1}(B) \in \mathcal{B}$

Therefore $\mathcal{C}$ is a σ-algebra that contains the open balls, thus $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{B}$ and we have equality which proves the desired result.

(1.5.b) If f is Lebesgue measurable, then so is F

proof Let $\mathcal{L}$ denote the set of Lebesgue measurable sets. Similarly, it suffices to show that for all $L \in \mathcal{L}$, $g^{-1}(L) \in \mathcal{L}$. For any Lebesgue measurable set $L \in \mathcal{L}$, there exists a Borel set $B \in \mathcal{B}$ and a null set $N \in \mathcal{L}$ such that

$$L = B \Delta N$$

and we have $g^{-1}(L) = g^{-1}(B) \Delta g^{-1}(N)$, thus it suffices to show that $g^{-1}(N)$ is a null set.

We have

$$\begin{aligned}
m(g^{-1}(N)) &= \int_{\mathbb{R}^2} 1_{g^{-1}(N)} dm \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{g^{-1}(N)}(x,y) dm(x) dm(y) && \text{by Fubini's theorem} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{N+y}(x) dm(x) dm(y) \\
&= \int_{\mathbb{R}} m(N+y) dm(y) \\
&= \int_{\mathbb{R}} m(N) dm(y) && \text{using the fact that translation preserves the Lebesgue measure} \\
&= \int_{\mathbb{R}} 0 dm(y) = 0
\end{aligned}$$

This completes the proof.

(1.6) Let $f : \mathbb{R}^2 \to \mathbb{R}$, and let $f_x(y) = f^y(x) = f(x,y)$. Suppose that for each x, f_x is Borel measurable, while for each y, f^y is continuous. Then f is Borel measurable.

proof For all $n \in \mathbb{N}$, let

$$\begin{aligned}
h_n &: \mathbb{R}^2 \to \mathbb{R} \\
(x,y) &\mapsto f(\lfloor nx \rfloor / n, y)
\end{aligned}$$

Then we have for all $c \in \mathbb{R}$,

$$h_n^{-1}([-\infty, c)) = \cup_{z \in \mathbb{Z}} \left([\frac{z}{n}, \frac{z+1}{n}) \times f_{\frac{z}{n}}^{-1}([-\infty, c)) \right)$$

which is a countable union of rectangles, therefore h_n is measurable.

Finally, we have

$$\begin{aligned}
\lim_n g_n(x,y) &= \lim_n f(\lfloor nx \rfloor / n, y) \\
&= \lim_n f^y(\lfloor nx \rfloor / n) \\
&= f^y(\lim_n \lfloor nx \rfloor / n) && \text{by continuity of } f^y \\
&= f^y(x) \\
&= f(x,y)
\end{aligned}$$

therefore f is Borel measurable as the limit of Borel measurable functions.

(1.7) Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be σ-algebras on spaces X, Y, Z respectively. Then

- $\phi((\mathcal{A} \times \mathcal{B}) \times \mathcal{C}) = \mathcal{A} \times (\mathcal{B} \times \mathcal{C})$ where

$$\begin{aligned} \phi : (X \times Y) \times Z &\to X \times (Y \times Z) \\ ((x,y),z) &\mapsto (x,(y,z)) \end{aligned}$$

  is a bijection.

- $\psi(\mathcal{A} \times \mathcal{B}) = \mathcal{B} \times \mathcal{A}$ where

$$\begin{aligned} \psi : X \times Y &\to Y \times X \\ (x,y) &\mapsto (y,x) \end{aligned}$$

  is a bijection.

proof we show that $\phi((\mathcal{A} \times \mathcal{B}) \times \mathcal{C})$ is a σ-algebra of $X \times (Y \times Z)$ that contains rectangles of the form $A \times M$ where $A \in \mathcal{A}$ and $M \in \mathcal{B} \times \mathcal{C}$.

We have

- for any subset $E \subset (X \times Y) \times Z$, $\phi(E)^c = \phi(E^c)$
- for any sequence of subsets (E_n), $\phi(\cup_n E_n) = \cup_n \phi(E_n)$

and since $(\mathcal{A} \times \mathcal{B}) \times \mathcal{C}$ is a σ-algebra, then so is its image by ϕ.

Now let us prove that it contains the required rectangles. Let $A \in \mathcal{A}$, and let

$$\mathcal{M} = \{M \in \mathcal{B} \times \mathcal{C} | A \times M \in \phi((\mathcal{A} \times \mathcal{B}) \times \mathcal{C})\}$$

we have

- $\mathcal{M}$ contains the rectangles of the form $B \times C$ where $B \in \mathcal{B}$ and $C \in \mathcal{C}$, since

$$A \times (B \times C) = \phi((A \times B) \times C)$$

  and $(A \times B) \times C \in (\mathcal{A} \times \mathcal{B}) \times \mathcal{C}$

- $\mathcal{M}$ is a monotone class: let $M_1 \subseteq M_2 \subseteq \ldots$ be an ascending sequence of elements of $\mathcal{M}$, and let $M = \cup_n M_n$. Then we have

$$A \times (\cup_n M_n) = \cup_n A \times M_n$$

  and for all n, $A \times M_n \in \phi((\mathcal{A} \times \mathcal{B}) \times \mathcal{C})$, which we proved is a σ-algebra. Therefore $A \times M \in \phi((\mathcal{A} \times \mathcal{B}) \times \mathcal{C})$. Similarly, $\mathcal{M}$ is closed under complement and countable descending intersection.

By the monotone class lemma, the smallest monotone class containing rectangles of the form $B \times C$, $B \in \mathcal{B}$, $C \in \mathcal{C}$, is a σ-algebra, which in turn contains the smallest such $\sigma-$algebra, i.e. $\mathcal{B} \times \mathcal{C}$. Therefore we have

$$\mathcal{B} \times \mathcal{C} \subseteq \mathcal{M} \subseteq \mathcal{B} \times \mathcal{C}$$

and we have equality. Therefore $\phi((\mathcal{A} \times \mathcal{B}) \times \mathcal{C})$ contains rectangles of the form $A \times M$ with $A \in \mathcal{A}$ and $M \in \mathcal{B} \times \mathcal{C}$. This proves

$$\mathcal{A} \times (\mathcal{B} \times \mathcal{C}) \subseteq \phi((\mathcal{A} \times \mathcal{B}) \times \mathcal{C})$$

similarly we can show that $\phi^{-1}(\mathcal{A} \times (\mathcal{B} \times \mathcal{C})) \subseteq (\mathcal{A} \times \mathcal{B}) \times \mathcal{C}$, which completes the proof.