MATH 202A - Problem Set 9

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(9.1) A collection of sets $\{E_\alpha\}_{\alpha \in A}$ has the countable intersection property if for every countable subset $A' \subseteq A$, $\cap_{\alpha \in A'} E_\alpha$ is non empty.

Let X be a topological space, and suppose that X is Lindelof. Let $\{E_\alpha\}_{\alpha \in A}$ be a collection of closed subsets that has the countable intersection property. Then $\cap_{\alpha \in A} E_\alpha \neq \emptyset$

proof We show the contrapositive: Assume that $\cap_{\alpha \in A} E_\alpha = \emptyset$. Then for all $x \in X$, there exists $\alpha_x \in A$ such that $x \notin E_{\alpha_x}$, i.e. $x \in E^c_{\alpha_x}$. This defines an open cover of X: $\{E^c_{\alpha_x}\}_{x \in X}$. Since X is Lindelof, this open cover has a countable subcover: there exists a sequence $(x_n)_n$ of elements of X such that $X = \cup_{i \in \mathbb{N}} E^c_{\alpha_{x_i}}$. Taking the complement, we have $\cap_{i \in \mathbb{N}} E_{\alpha_{x_i}} = \emptyset$. Therefore $\{E_\alpha\}_{\alpha \in A}$ does not have the countable intersection property.

(9.2) Let A, B be disjoint closed subsets of a normal space X. Then the following properties are equivalent:

1. There exists continuous $f : X \to [0, 1]$ such that $x \in B \Rightarrow f(x) = 1$ and $x \in A \Leftrightarrow f(x) = 0$.
2. A is a countable intersection of open sets.

proof

- $1 \Rightarrow 2$ Let $f : X \to [0, 1]$ be continuous such that $x \in B \Rightarrow f(x) = 1$ and $x \in A \Leftrightarrow f(x) = 0$. For all $n \in \mathbb{N}^*$, let $A_n = \{x \in X : f(x) < 1/n\} = f^{-1}[0, 1/n)$. Since f is continuous and $[0, 1/n)$ is open in the relative topology, A_n is open. And we have

  $$A = \cap_{n=1}^{\infty} A_n$$

  indeed, if $x \in A$, then $f(x) = 0$, thus $\forall n \geq 1$, $f(x) < 1/n$, i.e. $x \in A_n$. Conversely, if $x \in \cap_{n=1}^{\infty} A_n$, then $\forall n \geq 1$, $0 \leq f(x) < 1/n$, therefore $f(x) = 0$, i.e. $x \in A$.

  This proves that A is the countable intersection of open sets.

- $2 \Rightarrow 1$ Suppose A is the countable intersection of open sets A_n. For all $n \in \mathbb{N}$, let $V_n = A^c_n \cup B$. We have V_n is closed, and $A \cap V_n = \emptyset$ since $A \cap B = \emptyset$ and $A \cap A^c_n = \emptyset$. Thus A and V_n are closed disjoint subsets, and by Urysohn's lemma, there exists a continuous function $f_n : X \to [0, 1]$ such that $x \in V_n \Rightarrow f_n(x) = 1$, and $x \in A \Rightarrow f_n(x) = 0$.

  Now consider

  $$f : X \to [0, 1]$$
  $$x \mapsto f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x)$$

  Then we have

$$\begin{aligned} x \in B \Rightarrow \forall n,\ x \in V_n & \qquad\qquad \text{since } B \subseteq V_n \\ \Rightarrow \forall n,\ f_n(x) = 1 & \\ \Rightarrow f(x) = \sum_{n\geq 1} 2^{-n} = 1 & \end{aligned}$$

And if $x \notin A$, then there exists $n_0 \in \mathbb{N}$ such that $x \notin A_{n_0}$, i.e. $x \in A_{n_0}^c \subseteq V_{n_0}$. Therefore $f_{n_0}(x) = 1$, and

$$f(x) \geq 2^{-n_0} f_{n_0}(x) = 2^{-n_0} > 0$$

thus $x \notin A \Rightarrow f(x) \neq 0$. Conversely, if $x \in A$, then $\forall n,\ f_n(x) = 0$, thus $f(x) = 0$. This proves that $f(x) = 0 \Leftrightarrow x \in A$.

Finally, f is continuous: let $x \in X$, and let $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $2^{-N} < \epsilon/2$. Then we have for all $y \in X$

$$|f(y) - f(x)| \leq \sum_{n=1}^{N} 2^{-n}|f_n(y) - f_n(x)| + \sum_{n=N+1}^{\infty} 2^{-n}|f_n(y) - f_n(x)|$$

and

$$\begin{aligned} \sum_{n=N+1}^{\infty} 2^{-n}|f_n(y) - f_n(x)| &\leq \sum_{n=N+1}^{\infty} 2^{-n} \\ &\leq 2^{-N} \\ &< \epsilon/2 \end{aligned}$$

now for all $n \in \{1, \ldots, N\}$, f_n is continuous, thus there exists an open $U_n \subseteq X$ such that $x \in U_n$, and $\forall y \in U_n$, $|f_n(y) - f_n(x)| \leq \epsilon/(2N)$. Let $U = \cap_{n=1}^{N} U_n$. Then U is open, contains x, and for all $y \in U$,

$$\begin{aligned} |f(y) - f(x)| &< \sum_{n=1}^{N} 2^{-n}|f_n(y) - f_n(x)| + \epsilon/2 \\ &< \sum_{n=1}^{N} \epsilon/(2N) + \epsilon/2 \\ &< \epsilon \end{aligned}$$

This proves that f is continuous at y.

(9.3) A topological space X is said to be *completely regular* if X is T_1 and every closed set $E \subset X$ and $x \notin E$, there exists continuous $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for every $y \in E$.

Let $X = \prod_{s \in S} X_s$, where $\forall s \in S$, X_s is completely regular. Then X is completely regular.

proof X is T_1: let x, y be distinct elements in E. Then there exists $s \in S$ such that $x_s \neq y_s$. Since X_s is T_1, there exists an open subset $U_s \subseteq X_s$ such that $x_s \in U_s$ and $y_s \notin U_s$. Now consider $U = \times_{t \in S} U_t$ where $U_t = X_t$ for all $t \neq s$. Then U is open, $x \in U$, and $y \notin U$. This proves that X is T_1.

To prove that X is completely regular, let $E \subseteq X$ be a closed subset, and let $x \notin E$.

The complement E^c is open, therefore it is the union of basic open sets, $E^c = \cup_{\alpha \in A} U_\alpha$. And since $x \in E^c$, there exists $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$. U_{α_0} is a basic set, thus it is of the form

$$U_{\alpha_0} = \mathop{\times}_{s \in S} U_s$$

where U_s is open for all s, and $U_s = X_s$ for all but finitely many s. Let $\{s_1, \ldots, s_n\}$ be the set of elements of S such that $U_s \neq X_s$. Then we have

$$x \in U_{\alpha_0} \Leftrightarrow \forall i \in \{1, \ldots, n\},\ x_{s_i} \in U_{s_i}$$

For all $i \in \{1, \ldots, n\}$, we have $U_{s_i}^c$ is closed, and $x_i \notin U_{s_i}^c$. Thus since X_{s_i} is completely regular, there exists a continuous function $f_i : U_{s_i} \to [0,1]$ such that

$$\begin{aligned} f_i(x_{s_i}) &= 0 \\ \forall y_{s_i} \in U_{s_i}^c,\ f_i(y_{s_i}) &= 1 \end{aligned}$$

Now define

$$\begin{aligned} f : X &\to [0,1] \\ x &\mapsto \max_{i \in \{1,\ldots,n\}} f_i(x_{s_i}) \end{aligned}$$

then f is continuous as the pointwise maximum of finitely many continuous functions $x \mapsto f_i(x_{s_i})$ (composition of f_i with the projection on coordinate s_i, which is continuous). And we have

- For all $i \in \{1, \ldots, n\}$, by definition of f_i, $f_i(x_{s_i}) = 0$, thus $f(x) = \max_i f_i(x_{s_i}) = 0$
- For all $y \in E$, we have $y \notin E^c = \cup_{\alpha \in A} U_\alpha$, thus in particular $y \notin U_{\alpha_0}$, i.e. there exists $i_0 \in \{1, \ldots, n\}$ such that $y_{s_{i_0}} \notin U_{s_{i_0}}$. Therefore by definition of f_{i_0}, $f_{i_0}(y_{s_{i_0}}) = 1$, and we have

  $$f(y) = \max_i f_i(y_{s_i}) \geq f_{i_0}(y_{s_{i_0}}) = 1$$

  which proves $f(y) = 1$.

Hence f has the desired properties.

(9.4) Any separable metrizable space has a metrizable compactification.

proof Let $(X, \mathcal{T}^X)$ be a separable metrizable space, and let d be a distance such that $\mathcal{T}^X$ is the topology induced by d. Without loss of generality, assume that for all $x, y \in X$, $d(x,y) \leq 1$.

Since (X,d) is a separable metric space, there exists a countable dense set $D = \{y_n\}_{n \in \mathbb{N}}$. Let $Y = \prod_{n \in \mathbb{N}} [0,1]$, equipped with the product topology, denoted $\mathcal{T}^Y$. Consider the function

$$\begin{aligned} \phi : X &\to Y \\ x &\mapsto \phi(x) \end{aligned}$$

such that $\forall n \in \mathbb{N}$, $\phi_n(x) = d(x, y_n)$ (by assumption on d, $d(x,y_n) \in [0,1]$). Then we have

- ϕ is injective: if $\phi(x) = \phi(x')$, then $\forall n \in \mathbb{N}$, $\phi_n(x) = \phi_n(x')$, i.e. $d(x,y_n) = d(x', y_n)$. Now since $D = \{y_n\}_{n \in \mathbb{N}}$ is a dense subset of X, there exists a sequence of elements of D that converges to x. Let $(x_n)_n$ be such a sequence. Then we have for all $n \in \mathbb{N}$, $d(x, x_n) = d(x', x_n)$, thus $\lim_{n\to\infty} d(x', x_n) = \lim_{n\to\infty} d(x, x_n) = 0$, therefore (x_n) converges to x', and by uniqueness of the limit, $x = x'$.
- $\phi : X \to \phi(X)$ is surjective.
- ϕ is continuous: let $V \subset Y$ be an open subset of Y, and let $U = \phi^{-1}(V)$. Since V is open in the product topology, it is the union of basic open sets: $V = \cup_{\alpha \in A} V_\alpha$, where for each α, V_α is a basic open set. Then we have

  $$\begin{aligned} U &= \phi^{-1}(V) \\ &= \phi^{-1}(\cup_{\alpha \in A} V_\alpha) \\ &= \cup_{\alpha \in A} \phi^{-1}(V_\alpha) \end{aligned}$$

Therefore, to show that U is open, it suffices to show that the inverse image of any basic open is open. Let V_α be a basic open set. Then there exist open subsets of Y, $W_1, \ldots, W_k$, and indices $n_1, \ldots, n_k$ such that

$$y \in V_\alpha \Leftrightarrow \forall i \in \{1, \ldots, k\},\ y_{n_i} \in W_i$$

and we have

$$\begin{aligned} x \in \phi^{-1}(V_\alpha) &\Leftrightarrow \phi(x) \in V_\alpha \\ &\Leftrightarrow \forall i \in \{1, \ldots, k\},\ \phi_{i_k}(x) \in W_i \\ &\Leftrightarrow \forall i \in \{1, \ldots, k\},\ x \in \phi_{n_i}^{-1}(W_i) \\ &\Leftrightarrow x \in \cap_{i=1}^k \phi_{n_i}^{-1}(W_i) \end{aligned}$$

therefore $\phi^{-1}(V_\alpha) = \cap_{i=1}^k \phi_{n_i}^{-1}(W_i)$ is the finite intersection of open subsets (W_i is open and $\phi_{n_i} : x \mapsto d(x, y_{n_i})$ is continuous), thus is open. This proves that $U = \phi^{-1}(V)$ is open.

Now any open subset of $\phi(X)$ (equipped with the relative topology), is of the form $V \cap \phi(X)$ where V is open in Y, and we have $\phi^{-1}(V \cap \phi(X)) = \phi^{-1}(V) \cap \phi^{-1}(\phi(X)) = \phi^{-1}(V) \cap X = \phi^{-1}(V)$, which we proved is open. This proves that ϕ is continuous.

- the inverse of ϕ

$$\begin{aligned} \phi^{-1} : \phi(X) &\to X \\ y &\mapsto \phi^{-1}(y) = x \end{aligned}$$

is continuous. Indeed, let $x_0 \in X$, and let $U \subseteq X$ be an open neighborhood of x_0. We want to find an open set $V \subseteq Y$ (containing $\phi(X)$) such that $\phi^{-1}(V) \subseteq U$. Since $\{y_n\}_{n \in \mathbb{N}}$ is dense subset of X and U is open, U contains y_{n_0} for some n_0. And since $y_{n_0} \in U$ open, there exists $\epsilon > 0$ such that $B(\epsilon, y_{n_0}) \subseteq U$. Now let $V = \phi(X) \cap \times_{n=1}^\infty V_n$ where

$$\begin{aligned} &\forall n \neq n_0,\ V_n = [0, 1] \\ &V_{n_0} = [0, \epsilon) \end{aligned}$$

Then V is open in $\phi(X)$ ($\times_n V_n$ is a basic open set in Y), and $\forall y \in V$, let $x = \phi^{-1}(y)$, then we have $\phi(x) \in V$, thus $\phi_{n_0}(x) < \epsilon$, i.e. $d(x, y_{n_0}) < \epsilon$, i.e. $x \in B(\epsilon, y_{n_0})$, thus $x \in U$. This proves that $\phi^{-1}(V) \subseteq U$, and hence ϕ^{-1} is continuous.

Therefore $\phi : X \to \phi(X)$ is a homeomorphism.

Next, let $X^* = \mathrm{cl}(\phi(X))$. We have $(Y, \mathcal{T}^Y)$ is metrizable (countable product of compact sets). Therefore $(X^*, \mathcal{T}^*)$ is also metrizable, where $\mathcal{T}^*$ is the relative topology induced by $\mathcal{T}^Y$ on X^*, i.e. $\mathcal{T}^* = \mathcal{T}^Y_{|X^*}$. We also have X^* is a closed subset of the compact space Y, therefore it is compact.

$$(X, \mathcal{T}) \xrightarrow{\phi} (\phi(X), \mathcal{T}^Y_{|\phi(X)}) \subseteq (X^*, \mathcal{T}^*)$$

Therefore X is homeomorphic to $\phi(X)$, which is a dense subset of the compact metrizable X^*. To show that $(X^*, \mathcal{T}^*)$ is a compactification of X, we need to check that the topology of $\phi(X)$, $\mathcal{T}^Y_{|\phi(X)}$, is the relative topology induced by $\mathcal{T}^*$ on $\phi(X)$. In other words, we need to check that $\mathcal{T}^Y_{|\phi(X)} = \mathcal{T}^*_{|\phi(X)}$. This is true since

$$\begin{aligned} \mathcal{T}^*_{|\phi(X)} &= \{U \cap \phi(X), U \in \mathcal{T}^*\} \\ &= \{(V \cap X^*) \cap \phi(X), V \in \mathcal{T}^Y\} && \text{since } \mathcal{T}^* = \mathcal{T}^Y_{|X^*} \\ &= \{V \cap \phi(X), V \in \mathcal{T}^Y\} && \text{since } \phi(X) \subseteq X^* \\ &= \mathcal{T}^Y_{|\phi(X)} \end{aligned}$$

Therefore $(X^*, \mathcal{T}^*)$ is a metrizable compactification of X.