MATH 202A - Problem Set 8

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(8.1) Let $(X, \mathcal{T})$ be a compact topological space. Then each infinite subset of X has a limit point.

proof Let $A \subseteq X$ be a subset of X, and assume that A does not have a limit point, i.e. $\forall x \in X$, there exists an open neighborhood of x, $V_x \subset X$, such that $V_x \cap A = \{x\}$.

Let $\mathcal{V} = \{V_x\}_{x \in X}$. This is an open cover of X, and since X is compact, $\mathcal{V}$ has a finite subcover, $\{V_{x_i}\}_{i \in \{1,\dots,n\}}$. Thus we have

$$\begin{aligned} A &= A \cap X \\ &= A \cap (\cup_{i=1}^n V_{x_i}) \\ &= \cup_{i=1}^n (A \cap V_{x_i}) \\ &= \cup_{i=1}^n \{x_i\} \\ &= \{x_1, \dots, x_n\} \end{aligned}$$

therefore A is finite. This proves the contrapositive of the desired result.

(8.2) Let X be a compact Hausdorff space, and let $\sim$ be an equivalence relation on X such that $R = \{(x, y) \in X \times X | x \sim y\}$ is closed. Let

$$\begin{aligned} q : X &\to X/\sim \\ x &\mapsto q(x) \end{aligned}$$

be the quotient map (canonical surjection).

(8.2.a) For each $x \in X$, $q^{-1}(q(x))$ is closed.

proof

Remark 1 *If C_1 and C_2 are two closed subsets of X, then $C_1 \times C_2$ is closed in the product topology.*

proof The complement of $C_1 \times C_2$ in $X \times X$ is $X \times (X \setminus C_2) \cup (X \setminus C_1) \times X$, which is open as a union of two open sets in the product topology. Therefore $C_1 \times C_2$ is closed.

Remark 2 *The projection p_2*

$$\begin{aligned} p_2 : X \times X &\to X \\ (x, y) &\mapsto y \end{aligned}$$

is continuous

proof If O is an open subset of X, $p_2^{-1}(O) = X \times O$ which is open in the product topology.

Remark 3 *The projection p_2 is closed.*

proof Let A be a closed subset of $X \times X$. Since X is compact, the product $X \times X$ is compact. Thus A is compact as a closed subset of a compact set. And since p_2 is continuous, we have $p_2(A)$ is compact. Finally, since X is Hausdorff, any compact subset of X is closed, in particular $p_2(A)$ is closed.

Let

$$\begin{aligned} \phi : X &\to P(X) \\ x &\mapsto q^{-1}(q(x)) \end{aligned}$$

We have $\forall x \in X$

$$\begin{aligned} y \in \phi(X) &\Leftrightarrow q(y) = q(x) \\ &\Leftrightarrow y \sim x \\ &\Leftrightarrow (x, y) \in R \end{aligned}$$

Therefore

$$\begin{aligned} \phi(x) &= \{y \in X | (x, y) \in R\} \\ &= p_2(R \cap (X \times \{x\})) \end{aligned}$$

where p_2 is the projection on he second coordinate. We have

- $X \times \{x\}$ is closed since X is closed, $\{x\}$ is closed (the space is Hausdorff thus T_1) and the product of two closed subsets is closed by Remark 1
- $R \cap (X \times \{x\})$ is closed as the intersection of two closed subsets.
- p_2 is closed by Remark 3

Therefore $\phi(x) = p_2(R \cap (\{x\} \times X))$ is the image of a closed subset by a continuous function, thus is closed.

(8.2.b) Let $U \subseteq X$ be open. Then $\{x \in X | q^{-1}q(x) \subseteq U\}$ is open.

proof Let

$$\begin{aligned} \psi : P(X) &\to P(X) \\ U &\mapsto \psi(U) = \{x \in X | \phi(x) \subseteq U\} \end{aligned}$$

Let U be an open subset of X. Then we have

$$\begin{aligned} \psi(U)^c &= \{y \in X | \phi(y) \nsubseteq U\} \\ &= \{y \in X | \phi(y) \cap U^c \neq \emptyset\} \\ &= \{y \in X | \exists x \in U^c : (x, y) \in R\} \\ &= p_2(R \cap (U^c \times X)) \end{aligned}$$

We have $R \cap (U^c \times X)$ is closed since R is closed by assumption and $U^c \times X$ is closed by Remark 1. And since p_2 is closed (see 2.a), then $\psi(U)^c$ is closed, i.e. $\psi(U)$ is open.

(8.2.c) The compact quotient $X/\sim$ is Hausdorff.

proof Note: the quotient is compact since the quotient map is (by definition of the quotient topology) continuous, thus the quotient is the image of a compact space by a continuous function, hence is compact.

We have the following properties of ϕ and ψ:

1. $\forall x \in X$, $x \in \phi(x)$. This is obvious since $\phi(x)$ is the set of elements that are equivalent to x.

2. $\forall U \subseteq X$, $U \subseteq \phi(U)$. Follows from property 1.

3. $\forall U \subseteq X$, $\psi(U) \subseteq U$. Indeed, $\forall x \in \psi(U)$, $\phi(x) \subseteq U$, and $x \in \phi(x)$ (property 1), thus $x \in U$

4. $\forall U \subseteq X$, $\phi(\psi(U)) = \psi(U)$. Indeed, we have

   - $\psi(U) \subseteq \phi(\psi(U))$ by property 2
   - now let $x \in \phi(\psi(U))$. Then there exists $y \in \psi(U)$ such that $q(x) = q(y)$, thus $q^{-1}(q(x)) = q^{-1}(q(y))$. And $y \in \psi(U)$ is equivalent to $q^{-1}(q(y)) \subseteq U$. Therefore $q^{-1}(q(x)) \subseteq U$, which proves $x \in \psi(U)$. This proves the second inclusion $U \subseteq \psi(U)$.

Let $q(x) \neq q(y)$ be two distinct elements of the quotient space $X/\sim$, where $x, y \in X$ are representatives of $q(x)$, $q(y)$ respectively. By (8.2.a), the subsets $\phi(x) = q^{-1}(q(x))$ and $\phi(y) = q^{-1}(q(y))$ are closed. They are also disjoint since $q(x) \neq q(y)$.

Since X is compact and Hausdorff, it is normal, and we can separate $\phi(x)$ and $\phi(y)$: there exist disjoint open subsets $V_x, V_y \subseteq X$ such that $\phi(x) \subseteq V_x$ and $\phi(y) \subseteq V_y$. Now consider $q(\psi(V_x))$ and $q(\psi(V_y))$. We show that these sets separate $q(x)$ and $q(y)$. Indeed, we have

- $q^{-1}(q(\psi(V_x))) = \psi(V_x)$ by property 4, and $\psi(V_x)$ is open by (8.2.b). Therefore (by definition of the quotient topology), $q(\psi(V_x))$ is open. Similarly, $q(\psi(V_y))$ is open.

- $x \in \psi(V_x)$ since $\phi(x) \subseteq V_x$ by definition of V_x. Therefore $q(x) \in q(\psi(V_x))$. Similarly, $q(y) \in q(\psi(V_y))$.

- $q(\psi(V_x))$ and $q(\psi(V_y))$ are disjoint: first, $\psi(V_x)$ and $\psi(V_y)$ are clearly disjoint since a point z in their intersection must satisfy $\phi(z) \subseteq V_x$ and $\phi(z) \subseteq V_y$, which is not possible since V_x and V_y are disjoint. Then using property 4, $q^{-1}(q(\psi(V_x)))$ and $q^{-1}(q(V_y))$ are disjoint. Therefore, $q(\psi(V_x))$ and $q(\psi(V_y))$ are disjoint (the inverse images are disjoint only if the subsets are disjoint).

Finally, $q(\psi(V_x))$ and $q(\psi(V_y))$ are open disjoint subsets that separate $q(x)$ and $q(y)$.

(8.2.d) The quotient map q is closed (maps closed sets to closed sets)

proof Let A be a closed subset of the compact X. Then A is compact. And since q is continuous (by definition of the quotient topology), $q(A)$ is compact. Finally, since the quotient space $X/\sim$ is Hausdorff, $q(A)$ is closed (in a Hausdorff space, compact subsets are closed).

(8.2.e) The quotient map is not necessarily open.

proof A counter example is given by the set define in (2.f): indeed, consider the subset $A = [-\pi, 0)$. A is an open subset of X (with respect to the relative topology), but $q(A)$ is not an open subset of $X/\sim$, since $q^{-1}q(A) = [-\pi, 0) \cup \{\pi\}$, which is not open in X.

(8.2.f) Let $X = [-\pi, \pi]$, and let $S^1 = \{x \in \mathbb{C} | |x| = 1\}$ be the unit circle. Let $\sim$ be the equivalence relation given by $R = \{(x, x), x \in [-\pi, \pi]\} \cup \{(-\pi, \pi), (\pi, -\pi)\}$. Then $X/\sim$ is homeomorphic to S^1, and the quotient map can be taken to be

$$\begin{aligned} p : X &\to S^1 \\ x &\mapsto e^{ix} \end{aligned}$$

proof to show that the quotient space $X/\sim$ is homeomorphic to S^1, consider the function

$$f : X/\sim \to S^1$$
$$q(x) \mapsto e^{ix}$$

here x is any representative of the class $q(x)$. The function is well defined since if x, x' are two representatives of the same class, then $x = x' \mod (2\pi)$ and $e^{ix} = e^{ix'}$.

- The function f is injective since

$$\begin{aligned} f(q(x)) = f(q(x')) &\Rightarrow e^{ix} = e^{ix'} \\ &\Rightarrow x = x' \mod (2\pi) \\ &\Rightarrow q(x) = q(x') \end{aligned}$$

- It is surjective since $\forall y \in S^1$, there exists $x \in (-\pi, \pi]$ such that $y = e^{ix}$, and we have $f(q(x)) = e^{ix} = y$.

- It is continuous: let $y \in S^1$, and let $V = B(y, \epsilon) \cap S^1$ be an open neighborhood of y (note: any neighborhood of y contains a neighborhood of this form). Let $x \in [-\pi, \pi]$ such that $y = e^{ix}$. Then consider $U = \{q(x'), x' \in B(x, \eta) \mod (2\pi)\}$ where $\eta \in (0, \pi/2)$ satisfies $2\sin(\eta/2) < \epsilon$ (this is possible since $\sin(0) = 0$ and sin is continuous). U is open since $q^{-1}(U) = B(x, \eta) \mod (2\pi)$ is open (in the relative topology on $[-\pi, \pi]$). And we have for all $q(x') \in U$, $x' - x \mod (2\pi) \leq \eta$. Let $\delta = x' - x \mod (2\pi)$

$$\begin{aligned} |y - f(q(x'))| &= |f(q(x)) - f(q(x'))| \\ &= |e^{ix} - e^{i(x+\delta)}| \\ &= |e^{i(x-\delta/2)}||e^{i\delta/2} - e^{-i\delta}| \\ &= 2\sin(\delta/2) \\ &< \epsilon \end{aligned}$$

  therefore $f(q(x')) \in V$ for all $x' \in U$.

- Its inverse is continuous: let $q(x) \in X/\sim$, and let $U = \{q(x'), x' \in B(x, \eta) \mod (2\pi)\}$ $(\eta \in (0, \pi/2))$ be an open neighborhood of $q(x)$ (note: any neighborhood of of $q(x)$ contains a neighborhood of this form). Let $q(x) = f^{-1}(y)$, and let $V = B(y, \epsilon) \cap S^1$, where $\epsilon > 0$ is such that $\epsilon < 2\sin(\eta/2)$. Then we have V is an open neighborhood of y, and for all $y' \in V$, let $q(x') = f^{-1}(y')$. Then

$$|f(q(x')) - f(q(x))| < \epsilon$$

  i.e. $2\sin((x - x' \mod (2\pi))/2) < \epsilon < 2\sin(\eta/2)$, thus $x - x' \mod (2\pi) < \eta$. Therefore $x' \in U$, i.e. $f^{-1}(y') \in U$, which proves that f^{-1} is continuous.

(8.3) An open cover $\mathcal{U}$ of a topological space is locally finite if $\forall x \in X$, x has a neighborhood that intersects only finitely many members of $\mathcal{U}$. In particular, x lies in finitely many members of $\mathcal{U}$.

Suppose that $\mathcal{U}$ is a locally finite open cover of a normal space X. Then there exists a collection of open sets $\mathcal{V} = \{V_U, U \in \mathcal{U}\}$ such that

- $\mathcal{V}$ is open cover of X
- $\text{cl}(V_U) \subseteq U$ for all $U \in \mathcal{U}$

proof Consider S the set of all functions $F : \mathcal{U}_F \to P(X)$ where $\mathcal{U}_F \subseteq \mathcal{U}$ (the domain of F is a subfamily of $\mathcal{U}$), that satisfy the following conditions:

- $\forall U \in \mathcal{U}_F$, $F(U)$ is open, and $\text{cl}(F(U)) \subseteq U$
- $\{F(U)\}_{U\in\mathcal{U}_F} \cup \mathcal{U}_F^c$ (in $\mathcal{U}_F^c$, the complement is taken in $\mathcal{U}$) is an open cover of X, i.e.

$$\big(\cup_{U\in\mathcal{U}_F} F(U)\big) \cup \big(\cup_{V\in\mathcal{U}_F^c} V\big) = X$$

and equip S with the inclusion order: $F_1 \leq F_2$ if and only if $G_{F_1} \subseteq G_{F_2}$ where $G_F \subseteq \mathcal{U} \times P(X)$ is the graph of the function F, i.e. $G_F = \{(U, F(U)|U \in \mathcal{U}_F\}$.

Then we have:

- $S \neq \emptyset$, since S contains the empty function.
- Every chain in S has an upper bound. Indeed: let $(F_c)_{c\in C}$ be a chain in S. Define the function $F : \mathcal{U}_F \to P(X)$ to be the union of all functions, i.e.:
  - The domain of F is $\mathcal{U}_F = \cup_{c\in C}\mathcal{U}_{F_c}$. This domain satisfies $\mathcal{U}_F \subseteq \mathcal{U}$ since $\forall c, \mathcal{U}_{F_c} \subseteq \mathcal{U}$.
  - $\forall U \in \mathcal{U}_F$, there exists $c \in C$ such that $U \in \mathcal{U}_{F_c}$, then define $F(U)$ to be

$$F(U) = F_c(U)$$

    The function F is well defined, since if there are two elements $F_{c_1} \leq F_{c_2}$ in the chain such that $U \in \mathcal{U}_{F_{c_1}}$ and $U \in \mathcal{U}_{F_{c_2}}$, we must have $F_{c_1}(U) = F_{c_2}(U)$ since the graph of F_{c_1} is a subset of the graph of F_{c_2}.

  We have $F \in S$ since the following conditions are satisifed:

  - $\forall U \in \mathcal{U}_F$, $F(U)$ is open and $\text{cl}(F(U)) \subseteq U$, since $F(U) = F_c(U)$ for some F_c in the chain, and $F_c(U)$ satisfies $F_c(U)$ open and $\text{cl}(F_c(U)) \subseteq U$.
  - $\big(\cup_{U\in\mathcal{U}_F} F(U)\big) \cup \big(\cup_{V\in\mathcal{U}_F^c} V\big) = X$: indeed, let $x \in X$, and assume that $x \notin \cup_{V\in\mathcal{U}_F^c} V$. Then we show that $x \in \cup_{U\in\mathcal{U}_F} F(U)$. Since $\mathcal{U}$ is a locally finite open cover of X, x lies in finitely many members of $\mathcal{U}$, say $U_1, \ldots, U_n$ (where $n \geq 1$). And by assumption on x, $U_i \notin \mathcal{U}_F^c$ for all $i \in \{1, \ldots, n\}$, i.e. $U_i \in \mathcal{U}_F$.

    Now for all $i \in \{1, \ldots, n\}$, there exists $c_i \in C$ such that $U_i \in \mathcal{U}_{F_{c_i}}$. Consider the subchain $\{F_{c_i}\}_{i\in\{1,\ldots,n\}}$, and let F_m be its maximum (the maximum exists since the subchain is finite). For this maximum, we have $U_1, \ldots, U_n$ are all elements of the domain $\mathcal{U}_{F_m}$, therefore $x \notin \cup_{V\in\mathcal{U}_{F_m}^c} V$, and x must lie in $\cup_{U\in\mathcal{U}_{F_m}} F_m(U)$, i.e. there exists k such that $x \in F_m(U_k)$. Finally, since F is the union of all functions in the chain, we have $F(U_k) = F_m(U_k)$, and it follows that $x \in F(U_k)$, therefore $x \in \cup_{U\in\mathcal{U}_F} F(U)$, which completes the proof that F is an upper bound of the chain.

Applying Zorn's lemma to the set S, there exists a maximal element $f \in S$. First, we show that the domain of f is $\mathcal{U}$

$$\mathcal{U}_f = \mathcal{U}$$

by contradiction, assume this is not the case, and let $U_0 \in \mathcal{U} \setminus \mathcal{U}_f$. Then we construct a function $\tilde{f}$ such that $\tilde{f} > f$, which contradicts maximality of f:

Let

$$\begin{aligned}
\tilde{U}_0 &= (\cup_{U\in\mathcal{U}_f} f(U)) \cup (\cup_{V\in\mathcal{U}\setminus(\mathcal{U}_{\tilde{F}}\cup\{U_0\})} V) \\
E &= X \setminus U_0 \\
F &= X \setminus \tilde{U}_0
\end{aligned}$$

Since $f \in S$, we have $\tilde{U}_0 \cup U_0 = X$, therefore $X \setminus \tilde{U}_0 \subseteq U_0$, i.e. $F \subseteq U_0$.

Since E and F are closed (complements of open sets), and the space is normal, there exist open disjoint sets $V_E, V_F \subseteq X$ such that

$$E \subseteq V_E$$
$$F \subseteq V_F$$

and we have $V_F \subseteq (V_E)^c$ (V_E and V_F are disjoint), thus $\text{cl}(V_F) \subseteq (V_E)^c$ since $(V_E)^c$ is closed. But $(U_0)^c = E \subset V_E$, thus $(V_E)^c \subseteq U_0$. Therefore

$$\text{cl}(V_F) \subseteq U_0$$

Now define $\tilde{f}$ to be an extension of f, such that $\tilde{f}(U_0) = V_F$. Then we have $\tilde{f} > f$, and $\tilde{f} \in S$ since

- $\tilde{f}(U_0) = V_F$ is open
- $\text{cl}(\tilde{f}(U_0)) = \text{cl}(V_F) \subseteq U_0$
- We have

$$\begin{aligned} X &= \tilde{U}_0 \cup F && \text{by definition of } F \\ &\subseteq \tilde{U}_0 \cup V_F && \text{since } F \subseteq V_F \\ &\subseteq \tilde{U}_0 \cup \tilde{f}(U_0) \\ &= \left(\cup_{U \in \mathcal{U}_{\tilde{f}}} \tilde{f}(U) \right) \cup \left(\cup_{V \in \mathcal{U}_{\tilde{f}}^c} V \right) \end{aligned}$$

  which proves that we still have a cover of X.

This contradicts maximality of f, and completes the proof.

Using the fact the domain of f is $\mathcal{U}$, we have

$$X = \cup_{U \in \mathcal{U}} f(U)$$

where for each $U \in \mathcal{U}$, $f(U)$ is open and $\text{cl}(f(U)) \subseteq U$. This provides an open cover

$$\{f(U)\}_{U \in \mathcal{U}}$$

satisfying the required conditions.

(8.4) Let X be a normal topological space, and $\mathcal{U}$ be a locally finite open cover. Then there exists a family of continuous functions $\{f_U\}_{U \in \mathcal{U}}$ such that

- $f_U : X \to [0, 1]$
- $\forall x \in U^c$, $f_U(x) = 0$
- $\forall x \in X$, $\sum_{U \in \mathcal{U}} f_U(x) = 1$

proof First, construct, as in the previous problem, an open cover $\{V_U\}_{U \in \mathcal{U}}$ such that $\forall U \in \mathcal{U}$, V_U is open and $\text{cl}(V_U) \subseteq U$. Then for each $U \in \mathcal{U}$, consider the closed sets $\text{cl}(V_U)$ and U^c. These sets are disjoint since $\text{cl}(V_U) \subseteq U$. Thus by Urysohn's Lemma, there exists a continuous function $h_U : X \to [0, 1]$ such that $\forall x \in \text{cl}(V_U)$, $h_U(x) = 1$, and $\forall x \in U^c$, $h_U(x) = 0$. This defines a family of functions $\{h_U\}_{U \in \mathcal{U}}$. Now consider

$$\begin{aligned} h : X &\to \mathbb{R}_+ \\ x &\mapsto \sum_{U \in \mathcal{U}} h_U(x) \end{aligned}$$

this function is well defined since $\mathcal{U}$ is locally finite: $\forall x \in X$, x lies in finitely many members of $\mathcal{U}$, thus $\sum_{U \in \mathcal{U}} h_U(x)$ is finite.

We also have h is continuous: let $x \in X$, and let $\epsilon > 0$. Since $\mathcal{U}$ is locally finite, x has a neighborhood N_0 that intersects finitely many members of $\mathcal{U}$, say $U_1, \ldots, U_n$. In particular, $\forall y \in N_0$, $h(y) = \sum_{i=1}^n h_{U_i}(y)$ (all other terms in the sum are zero). For each $i \in \{1, \ldots, n\}$, since h_{U_i} is continuous, there exists an open neighborhood N_i of x such that $\forall y \in N_i$, $|h_{U_i}(y) - h_{U_i}(x)| < \epsilon/n$. Now let $N = \cap_{i=0}^n$. N is an open neighborhood of x, and for all $y \in N$ we have

$$\begin{aligned}
|h(y) - h(x)| &= |\sum_{i=1}^n (h_{U_i}(y) - h_{U_i}(x))| \\
&\leq \sum_{i=1}^n |h_{U_i}(y) - h_{U_i}(x)| \\
&\leq \sum_{i=1}^n \epsilon/n \\
&= \epsilon
\end{aligned}$$

which proves that h is continuous at x.

Furthermore, since $\{V_U\}_{U \in \mathcal{U}}$ is an open cover of X, $h(x) \geq 1$ for all $x \in X$ (indeed, for each x, there exists $U \in \mathcal{U}$ such that $x \in V_U$, and $h(x) \geq h_U(x) = 1$ since h_U is, by definition, identically 1 on $\mathrm{cl}(V_U)$). Now for each $U \in \mathcal{U}$, let

$$\begin{aligned}
f_U : X &\to [0, 1] \\
x &\mapsto h_U(x)/h(x)
\end{aligned}$$

f_U is continuous (h_U and h are continuous, h is positive), and satisfies $\forall x \in U^c$, $f_U(x) = h_U(x)/h(x) = 0$. And we have $\forall x \in X$, $\sum_{U \in \mathcal{U}} f_U(x) = 1$.

(8.5) Let X be a set with at least two elements, and that has a total ordering $\leq$ satisfying

- $x \leq y$ and $y \leq x$ imply $x = y$
- for any $x, y \in X$, either $x \leq y$ or $y \leq x$

Notation:

- Define $x < y$ iff $x \leq y$ and $x \neq y$.
- $\{x | x < a\}$ will be denoted $(-, a)$
- $\{x | a < x\}$ will be denoted $(a, +)$
- $\{x | a < x < b\}$ will be denoted (a, b)
- $\{x | a \leq x \leq b\}$ will be denoted $[a, b]$

Consider the order topology on X, for which a base is given by $\{(-, b), b \in X\} \cup \{(a, +), a \in X\} \cup \{(a, b), a, b \in X\}$

Then every open interval (a, b) is open and every closed interval $[a, b]$ is closed.

proof Every open interval (a, b) is open by definition. Every closed interval $[a, b] = \{x | a \leq x \leq b\}$ is the complement of the open subset $\{x | x < a\} \cup \{x | x > b\}$ (using the fact that $(a \not\leq b) \Leftrightarrow (b < a)$[1]) thus is closed.

[1] If $a \not\leq b$, since the ordering is total, we must have $b \leq a$, and since the ordering is reflexive, we must have $b \neq a$, therefore $b < a$

(8.6) X equipped with the order topology is Hausdorff and regular.

proof X **is Hausdorff**: let $x \neq y \in X$, and assume without loss of generality that $x < y$. Then consider the following cases:

1. If there exists a such that $x < a < y$, then the subsets $U_x = (-, a)$ and $U_y = (a, -)$ are open (by definition), disjoint, and $x \in U_x$ and $y \in U_y$, thus U_x and U_y separate x and y.

2. If there exists no a such that $x < a < y$, then the subsets $U_x = (-, y)$ and $U_y = (x, -)$ are open (by definition), disjoint (by assumption, there exists no a such that $x < a$ and $a < y$, i.e. there exists no a that belongs to both U_x and U_y) and $x \in U_x$ and $y \in U_y$ (since $x < y$). Therefore U_x and U_y separate x and y

X **is regular**: let $x \in X$, and let E be a closed subset of X that does not contain x. Since E^c is open, it is the union of basic open sets. And since $x \in E^c$, there exists a basic open set U such that $x \in U \subseteq E^c$. Now consider the following cases:

1. If there exist $a, b \in U$, such that $a < x < b$, then we have $x \in (a, b)$, and $[a, b] \subseteq U \subseteq E^c$, thus $E \subseteq [a, b]^c$. (a, b) and $[a, b]^c$ are open, disjoint, and $x \in (a, b)$ and $E \subseteq [a, b]^c$. Therefore (a, b) and $[a, b]^c$ separate x and E.

2. If there exists $a \in U$, such that $a < x$, and there exists no $b \in U$ such that $x < b$, then we have $x \in (a, -)$, and $[a, -) \subseteq U \subseteq E^c$, thus $E \subseteq [a, -)^c = (-, a)$. Therefore $(-, a)$ and $(a, -)$ are open, disjoint, $x \in (a, -)$ and $E \subseteq (-, a)$, thus $(a, -)$ and $(-, a)$ separate x and E.

3. If there exists $b \in U$, such that $x < b$, and there exists no $a \in U$ such that $a < x$, then we have (similarly to the previous case), $(b, -)$ and $(-, b)$ separate E and x.

4. If there exist no $a \in U$, such that $a < x$ and no $b \in U$ such that $x < b$, then $U = \{x\}$, thus U is open (by definition of U) and closed (since $U = [x, x]$). Thus we have U and U^c are open, disjoint, $x \in U$ and $E \subseteq U^c$. Therefore U and U^c separate x and E.