MATH 202A - Problem Set 5

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(5.1) The contraction mapping principle Let $0 \leq r < 1$. Let (X, d) be a nonempty metric space and $f : X \to X$ be a function that is a strict contraction, that is, for all $x, y \in X$, $d(f(x), f(y)) \leq rd(x, y)$. Then f has a unique fixed point.

proof Existence of a fixed point: Let $x_0 \in X$, and define the sequence $(x_n)_n$ by: for all $n \in \mathbb{N}$, $x_{n+1} = f(x_n)$. Then $(x_n)_n$ is a Cauchy sequence: first, we have by induction on $n \in \mathbb{N}$

$$d(x_{n+1}, x_n) \leq r^n d(x_1, x_0)$$

this is true for $n = 0$, and if it is true for n, then $d(x_{n+2}, x_{n+1}) = d(f(x_{n+1}), f(x_n)) \leq rd(x_{n+1}, x_n) \leq r.r^n d(x_1, x_0)$, which completes the induction.

Now we have for all $n \in \mathbb{N}$ and $k \in \mathbb{N}$,

$$\begin{aligned} d(x_{n+k}, x_n) &\leq \sum_{i=0}^{k-1} d(x_{n+i+1}, x_{n+i}) \\ &\leq \sum_{i=0}^{k-1} r^{n+i} d(x_1, x_0) \\ &\leq r^n d(x_1, x_0) \sum_{i=0}^{k-1} r^i \\ &\leq r^n \frac{d(x_1, x_0)}{1 - r} \end{aligned}$$

using the fact that $\sum_{i=0}^{k-1} r^i = \frac{1-r^k}{1-r} \leq \frac{1}{1-r}$. Let $\epsilon > 0$. Since $r^n \frac{d(x_1,x_0)}{1-r} \to 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, for all $k \geq 0$, $d(x_n, x_{n+k}) \leq \epsilon$. Therefore $(x_n)_n$ is a Cauchy sequence.

Since X is complete, $(x_n)_n$ converges. Let x be its limit. Since f is continuous and (x_n) converges to x, we have $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$, which proves that x is a fixed point.

Uniqueness of the fixed point. Let x and x' be two fixed points. Then we have

$$d(x, x') = d(f(x), f(x')) \leq rd(x, x')$$

therefore $d(x, x')(1 - r) \leq 0$, but since $1 - r > 0$, and $d(x, x') \geq 0$, we must have $d(x, x') = 0$, i.e. $x = x'$.

(5.2) Let $a \in \mathbb{C}$ and $h : \mathbb{C} \to \mathbb{C}$ be a bounded Lipschitz continuous function. Consider the differential equation

$$\frac{df}{dx} = h(f(x))$$

with the initial condition $f(0) = a$. Then there exists $\delta > 0$ and $f : (-\delta, \delta) \to \mathbb{C}$ solution of the problem.

proof It suffices to show that there exists a continuous functions from $f : [-\delta, \delta] \to \mathbb{C}$ (for some δ) that satisfies the integral equation

$$f(x) = a + \int_0^x h(f(t))dt, \forall x \in [-\delta, \delta]$$

Let $K > 0$ be a Lipschitz constant for h, and let $\delta > 0$ (that will be fixed later). Consider the space $X = C([-\delta, \delta], \mathbb{C})$ of continuous functions on $[-\delta, \delta]$, with the metric

$$\begin{aligned} d : X \times X &\to \mathbb{R} \\ (f, g) &\mapsto \sup_{x \in [\delta, \delta]} |f(x) - g(x)| \end{aligned}$$

The metric space (X, d) is complete. Consider the map

$$\begin{aligned} \phi : X &\to X \\ f &\mapsto \phi(f) = a + \int_0^x h(g(t))dt \end{aligned}$$

For all $f, g \in X$, we have

$$\begin{aligned} d(\phi(f), \phi(g)) &= \sup_{x \in [-\delta, \delta]} |\phi(f)(x) - \phi(g)(x)| \\ &= \sup_{x \in [-\delta, \delta]} |\int_0^x h(f(t)) - h(g(t))dt| \\ &\leq \sup_{x \in [-\delta, \delta]} \int_0^x |h(f(t)) - h(g(t))|dt \\ &\leq \int_{-\delta}^{\delta} |h(f(t)) - h(g(t))|dt \\ &\leq \int_{-\delta}^{\delta} K|f(t)) - g(t)|dt && \text{since } h \text{ is } K\text{-Lipschitz} \\ &\leq K \int_{-\delta}^{\delta} d(f, g)dt && \text{since } d(f, g) = \sup_{u \in [-\delta, \delta]} |f(u)) - g(u)| \\ &\leq 2K\delta d(f, g) \end{aligned}$$

Now if we choose in particular $\delta \in (0, 1/2K)$, we have ϕ is a contraction, since $2K\delta < 1$. By the contraction mapping theorem, ϕ has a unique fixed point $f \in C([-\delta, \delta], \mathbb{C})$, that satisfies

$$f = \phi(f) = a + \int_0^x h(f(t))dt$$

This completes the proof.

(5.4) Let X be a connected metric space and $f : X \to \mathbb{Z}$ be a continuous function. Then f is constant.

proof Let $x_0 \in X$ be any element of X, and $y_0 = f(x_0)$. Now consider $Y_1 = \mathbb{Z} \setminus \{y_0\}$, and let $X_0 = f^{-1}(\{y_0\})$ and $X_1 = f^{-1}(Y_1)$ (X_0 is nonempty since it contains x_0). Then we have

$$X = X_0 \cup X_1$$

since for all $x \in X$, $f(x)$ is either y_0 or an element of Y_1, thus x is either in X_0 or X_1. The sets X_0 and X_1 are disjoint since $\{y_0\}$ and Y_1 are disjoint by definition. Finally, since $\{y_0\}$ is open (the open ball $B(1/2, y_0)$ of $\mathbb{Z}$ is contained in $\{y_0\}$) and f is continuous, X_0 is open in X; and since Y_1 is open (for any $y \in Y_1$, the open ball $B(1/2, y)$ of $\mathbb{Z}$ is contained in Y_1) and f is continuous, X_1 is open.

Thus $X = X_0 \cup X_1$ is the union of two disjoint open sets. Since X is connected, at least one of the sets is empty, but since X_0 is nonempty, we must have $X_1 = \emptyset$, therefore

$$X = X_0 = f^{-1}(\{y_0\})$$

which proves that f is constant.

(5.5) Let (X, d) be a metric space that is both connected and locally pathwise connected. Then X is pathwise connected.

proof Let $x_0 \in X$, and consider the set S of open, pathwise connected subsets containing x_0. Since X is locally pathwise connected, S is nonempty. Every chain in $(S, \subseteq)$ has an upper bound (the union of all elements in the chain is still an element of S since the union of open sets is open, and the union of pathwise connected sets having a common element is pathwise connected[1]). Therefore, by Zorn's lemma, S has a maximal element. Let U be this maximal element (the largest open pathwise connected subset containing x_0). Then $U = X$. To show this, assume by contradiction that U^c is nonempty. Since X is connected, U may not be open (otherwise $X = U \cup U^c$ would be the union of two nonempty open subsets), thus there exists $x_1 \in U^c$ such that $\forall \epsilon > 0$, $B(\epsilon, x_1) \cap U$ is nonempty (i.e. x_1 is on the boundary of U). Now since X is locally pathwise connected, there exists an open, pathwise connected subset V containing x_1. Since V is open, there exists $\epsilon > 0$ such that $B(\epsilon, x_1) \subseteq V$. Now let $x_2 \in B(\epsilon, x_1) \cap U$ (the intersection is nonempty by definition of x_1). Then $U \cup B(\epsilon, x_1)$ is pathwise connected (both U and $B(\epsilon, x_1)$ are open and pathwise connected, and they have x_1 in common, therefore their union is ope and pathwise connected). But this contradicts maximality of U since $U \subset U \cup B(\epsilon, x_1)$ (we have strict inclusion since $x_1 \notin U$).

(5.6) Consider the sequence of polynomial functions $P_n : [0,1] \to \mathbb{R}$, defined by

$$P_0(x) = 0$$
$$P_{n+1}(x) = P_n(x) + \frac{1}{2}(x - P_n(x)^2)$$

Then for all $x \in [0,1]$, and for all $n \in \mathbb{N}$, $P_n(x) \le P_{n+1}(x)$, and $\lim_{n\to\infty} P_n(x) = \sqrt{x}$.

proof Let $x \in [0,1]$. We prove by induction that $P_{n+1}(x) \ge P_n(x)$.

Now let $n \in \mathbb{N}$. Since $P_{n+1}(x) \ge P_n(x)$, we have $P_n(x) + \frac{1}{2}(x - P_n(x)^2) \ge P_n(x)$, i.e. $x - P_n(x)^2 \ge 0$, i.e. $P_n(x) \le \sqrt{x}$. Therefore the sequence $(P_n(x))_n$ is non-decreasing and bounded above by $\sqrt{x}$, thus converges. Let l be its limit. Then taking the limit in the expression $P_{n+1}(x) = P_n(x) + \frac{1}{2}(x - P_n(x))^2$, we have $l = l + \frac{1}{2}(x - l^2)$. Therefore $l = \sqrt{x}$, and $(P_n(x))_n$ converges to $\sqrt{x}$.

(5.7) $x \mapsto \sqrt{x}$ is the uniform limit of polynomial functions on $[0,1]$.

proof Let $X = C([0,1], \mathbb{R})$ be the space of continuous real-valued functions on $[0,1]$, and $d : X \times X \to \mathbb{R}$ be the metric $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$. Then consider the complete metric space (X, d).

We have for all $x \in [0,1]$, $\sqrt{x} = \lim_{n\to\infty} P_n(x)$, and $\forall n \in \mathbb{N}$, $P_{n+1}(x) \le P_n(x)$, thus by (4.4), $x \mapsto \sqrt{x}$ is the uniform limit of the functions $(P_n)_n \in X$, which are by definition polynomial.

(5.8) Let (X, d) be a metric space which has no isolated points. Then X is uncountable.

proof by Baire's theorem Let $S = \{x_n, n \in \mathbb{N}\}$ be any countable subset of X. For all $n \in \mathbb{N}$, define $X_n = X \setminus \{x_n\}$. We have

- X_n is open since its complement $\{x_n\}$ contains a single element and is closed.
- X_n is dense since x_n is not an isolated point ($\inf x \neq x_n d(x, x_n) = 0$), therefore for all $\epsilon > 0$, there exists $x \neq x_n$ such that $d(x, x_n) \le \epsilon$, and since $x \neq x_n$, $x \in X_n$, thus $B(x_n, \epsilon)$ contains an element of

[1]Proof: let $E_{\alpha\alpha\in A}$ be a family of pathwise connected subsets that have a common element x_0. And let $E = \cup_{\alpha\in A} E_\alpha$. Let $x, x' \in E$. Then there exists $\alpha, \alpha' \in A$ such that $x \in E_\alpha$ and $x' \in E_{\alpha'}$. Since E_α is connected and contains both x and x_0, there exists a path $p : [0,1] \to E_\alpha$ such that $p(0) = x$ and $p(1) = x_0$. And since $E_{\alpha'}$ is pathwise connected and contains both x' and x_0, there exists a path $p' : [0,1] \to E_{\alpha'}$ such that $p'(0) = x'$ and $p'(1) = x_0$. Then the function $\tilde{p}$ defined as

$$\tilde{p}[0,2] \to E$$
$$t \mapsto \tilde{p}(t) = \begin{cases} p(t) & \text{if } t \in [0,1] \\ p'(2-t) & \text{if } t \in [1,2] \end{cases}$$

is a path that connects x to x' ($\tilde{p}$ is continuous and $\tilde{p}(0) = x$, $\tilde{p}(2) = x'$), therefore E is pathwise connected.

X_n for every $\epsilon > 0$, and x_n is in the closure of X_n, which proves that $X \subseteq X_n^{cl} \subseteq X$, hence $X_n^{cl} = X$ and X_n is dense in X.

By Baire's theorem, the countable intersection of open dense sets is nonempty, therefore $\cap_{n\in\mathbb{N}} X_n$ is non empty, i.e. S^c is nonempty (we have $S^c = \cup_{n\in\mathbb{N}}\{x_n\} = \cap_{n\in\mathbb{N}}\{x_n\}^c = \cap_{n\in\mathbb{N}} X_n$). Therefore X contains at least one element that is not in S, which proves that it is not countable.

a second proof Let $S = \{x_n, n \in \mathbb{N}\}$ be any countable set of elements of X. Then let us construct a sequence of nested nonempty open balls $E_n = B(r_n, y_n)$, $n \geq 1$, such that

- $y_n \neq x_n$ and $0 < r_n \leq \min(1/n, d(x_n, y_n)/2)$
- $E_{n+1}^{cl} \subseteq E_n$ for all $n \geq 1$

Let $y_1 \neq x_1$ (such an element exists, otherwise X would be reduced to a single element, which is isolated), and let E_1 be the open ball $E_1 = B(r_1, y_1)$ where $r_1 = \min(1, d(x_1, y_1)/2)$. Now suppose the open ball $E_n = B(r_n, y_n)$ constructed, and let us construct E_{n+1}. Since x_{n+1} is not isolated, we can find an element $y_{n+1} \in E_n$ such that $y_{n+1} \neq x_{n+1}$ (consider two cases: (i) if $x_{n+1} \neq y_n$, then take $y_{n+1} = y_n$. (ii) if $x_{n+1} = y_n$, since y_n is not isolated, there exists $y_{n+1} \neq y_n$ such that $d(y_n, y_{n+1}) \leq r_n/2$, thus we have $y_{n+1} \in B(y_n, r_n) = E_n$). Since $y_{n+1} \in E_n$ and E_n is open, there exists $\epsilon > 0$ such that $B(\epsilon, y_{n+1}) \subseteq E_n$. Let $r_{n+1} = \min(\epsilon/2, 1/(n+1), d(x_{n+1}, y_{n+1})/2)$, and $E_{n+1} = B(r_{n+1}, y_{n+1})$. Then E_{n+1} satisfies the desired properties since

- $y_{n+1} \neq x_{n+1}$ by definition. $r_{n+1} > 0$ as the minimum of three positive reals, and by definition $r_{n+1} \leq \min(1/(n+1), d(x_{n+1}, y_{n+1}))$.
- $E_{n+1}^{cl} \subseteq E_n$ since $E_{n+1}^{cl} \subseteq B(2r_{n+1}, y_{n+1}) \subseteq B(\epsilon, y_{n+1}) \subseteq E_n$.

This defines the sequence of nested open sets (E_n). The sequence (y_n) of the centers is a Cauchy sequence since for all $m \geq n$, y_m is an element of $E_m \subseteq E_n = B(r_n, y_n)$, thus

$$d(y_n, y_m) \leq r_n \leq 1/n$$

and converges to 0. Since X is complete, the Cauchy sequence $(y_n)_n$ converges. Let y be its limit. Then we have for every $N \in \mathbb{N}$, for all $n \geq N+1$, $y_n \in E_n \subseteq E_{N+1}$, therefore $(y_n)_{n\geq N+1}$ is a sequence of elements of E_{N+1}, and its limit $y \in E_{N+1}^{cl} \subseteq E_N$, therefore

$$d(y, x_N) \geq d(x_N, y_N) - d(y_N, y) \geq r_N > 0$$

using the fact that $d(x_N, y_N) \geq 2r_N$ (by definition of r_N) and $d(y_N, y) \leq r_N$ (since $y_N \in E_N^{cl}$). This proves that $y \neq x_N$ for any N, therefore $y \in S$, and X is not countable.

(5.9) Let $G = \{(x, \sin(1/x)), x \in (0,1]\} \cup \{(0, y), y \in [-1, 1]\}$. Then G is not pathwise-connected in $\mathbb{R}^2$.

proof Consider the metric

$$\begin{aligned} d : \mathbb{R}^2 \times \mathbb{R}^2 &\to \mathbb{R} \\ (a, b) &\mapsto |a_1 - b_1| + |a_2 - b_2| \end{aligned}$$

Assume, by contradiction, that there exists a path $p : [0,1] \to G$ connecting $(1/\pi, 0)$ and $(0,0)$ (p is continuous, $p(0) = (1/\pi, 0)$ and $p(1) = (0,0)$). Let

$$\begin{aligned} G_1 &= \{(x, \sin(1/x)), x \in (0,1]\} \\ G_2 &= \{(0, y), y \in [-1, 1]\} \end{aligned}$$

Now let

$$t_0 = \sup\{t \in [0,1] | p(t) \in G_1\}$$

($t_0 \geq 0$ since $p(0) \in G_1$). Let $\epsilon \in (0, 1)$. Since p is continuous, there exists $\eta > 0$ such that for all $t, t' \in [0, 1]$, if $|t - t'| \leq \eta$ then $d(p(t) - p(t')) \leq \epsilon$. By definition of t_0, there exists $\delta \in (0, \eta/2)$ such that $p(t_0 - \delta) \in G_1$, and we have $p(t_0 + \delta) \in G_2$. Consider the image of $[t_0 - \delta, t_0 + \delta]$ by p, $U = p([t_0 - \delta, t_0 + \delta])$. Let $p(t_0 - \delta) = (x_0, \sin(1/x_0))$. Then U contains $\{(x, \sin(1/x), x \in (0, x_0])\}$ [2] and in particular, U contains the points $(1/(\pi/2 + 2k\pi), -1)$ and $(-\pi/2 + 2k\pi, 1)$ for any k satisfying $-\pi/2 + 2k\pi > 1/x_0$. Therefore there exist $t_1, t_2 \in [t_0 - \delta, t_0 + \delta]$ such that $p(t_1) = (1/(\pi/2 + 2k\pi), -1)$ and $p(t_2) = (-\pi/2 + 2k\pi, 1)$. Then $|t_1 - t_2| \leq 2\delta \leq \eta$, yet $d(p(t_1) - p(t_2)) \geq 2 > \epsilon$. This leads to a contradiction and completes the proof.

[2] otherwise, there exists $x_1 \in (0, x_0)$ such that $(x_1, \sin(x_1)) \notin U$, and we can write U as the intersection $U = (U \cap L) \cup (U \cap R)$ where

$$L = (-1, x_1) \times (-2, 2)$$
$$R = (x_1, 2) \times (-2, 2)$$

We have

- L and R are open
- $U \cap R$ is nonempty since it contains $(x_0, \sin(1/x_0))$
- $U \cap L$ is nonempty sine L contains G_2, thus contains $p(t_0 + \delta) \in U$.

Thus U is not connected. But U is the image of a connected subset by a continuous function, thus is connected. Contradiction.