MATH 202A - Problem Set 4

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(4.1) Let E, F be nonempty subsets of $\mathbb{R}^n$, E is compact and F is closed. Then there exist $(e, f) \in E \times F$ such that $d(E, F) = d(e, f)$.

proof Let $\alpha = d(E, F) = \inf_{(x,y)\in E\times F} d(x, y)$ be the distance between the two sets. Let $\epsilon > 0$, and let x_0 be a given element of E. Since E is compact, it is in particular bounded, and there exists $r > 0$ such that $E \subset B(r, x_0)$. Now consider the closed ball $B(r + \alpha + \epsilon, x_0)^{cl}$ and let

$$F' = F \cap B(r + \alpha + \epsilon, x_0)^{cl}$$

We have F' is a closed and bounded subset of $\mathbb{R}^n$, thus it is compact. We also have

$$d(E, F) = \inf_{(x,y)\in E\times F'} d(x, y)$$

Indeed, for all $y \in F \setminus F'$ and for all $x \in E$

$$d(y, x) \geq d(y, x_0) - d(x, x_0)$$

by the triangle inequality. And since $d(y, x_0) \geq r + \alpha + \epsilon$, and $d(x, x_0) \leq r$ $(E \subset B(r, x_0))$, we have

$$d(y, x) \geq r + \alpha + \epsilon - r = \alpha + \epsilon$$

Therefore $\inf_{x\in E, y\in F\setminus F'} d(x, y) \geq \alpha + \epsilon > d(E, F)$ which proves the result.

Now from the equality $d(E, F) = \inf_{(x,y)\in E\times F'} d(x, y)$, we have that $d(E, F)$ is the infimum of the continuous function

$$\begin{aligned} E \times F' &\to \mathbb{R} \\ (x, y) &\mapsto d(x, y) \end{aligned}$$

on the compact set $E \times F'$. Thus the infimum is achieved, and there exist $(e, f) \in E \times F' \subseteq E \times F$ such that $d(e, f) = d(E, F)$.

(4.2) Let $E_1 \supset E_2 \supset E_3 \supset \dots$ be a nested sequence of bounded closed subsets of $\mathbb{R}^n$. Let $U \in \mathbb{R}^n$ be open. Then if $\cap_{k=1}^{\infty} E_k \subset U$, then there exists N such that $E_N \subset U$.

proof by contrapositive. Assume that there exists no n such that $E_n \subset U$. Then $\forall n \in \mathbb{N}$, there exists $x_n \in E_n$ such that $x_n \notin U$. This defines a sequence $(x_n)_n$ of elements of E_1 (since for all $n \in \mathbb{N}$, $E_n \subset E_1$). Since E_1 is a closed bounded subset of $\mathbb{R}^n$, it is compact, therefore $(x_n)_n$ has a converging subsequence $(x_{\phi(n)})_n$. Let x be its limit. Then we have for all $N \in \mathbb{N}$, $x \in E_N$. Indeed, let N' be such that $\phi(N') \geq N$. Then we have for all $n \geq N'$, $x_{\phi(n)} \in E_{\phi(n)} \subset E_{\phi(N')} \subset E_N$ $((x_{\phi(n)})_n$ is eventually in $E_N)$. Therefore $\lim_{n\to x_{\phi(n)}\in E_N^{cl}=E^N\infty}$ since E_N is closed by assumption. Thus we have $x \in \cap_{N=1}^{\infty} E_N$.

Now since, x is the limit of the sequence $(x_{\phi(n)})_n$ of elements of U^c, which is closed (U is open by assumption), we have $x \in U^c$. This shows that $x \in \cap_{N=1}^{\infty} E_N$ but $x \notin U$, which proves the contrapositive.

(4.3) Let (E_j) be a nested sequence of compact subsets of a metric space, $E_{j+1} \subset E_j$ for all j, and let $E = \cap_{j=1}^{\infty} E_j$. Let $\rho > 0$, and suppose that for all j, $d(E_j) \geq \rho$. Then $d(E) \geq \rho$. Here $d(S) = \sup_{s,t \in S} d(s,t)$ is the diameter of the set S.

proof Let $\epsilon > 0$. We have for all $j \in \mathbb{N}$, $\sup_{x,t \in E_j} d(s,t) \geq \rho$, thus $\exists s_j, t_j \in E_j$ such that $d(s_j, t_j) \geq \rho - \epsilon$. This defines two sequences of elements $(s_j)_j$ and $(t_j)_j$. We observe that since for all $j \in \mathbb{N}$, $E_j \subset E_1$, $(s_j)_j$ is a sequence of elements of the compact set E_1, therefore has a converging subsequence $(s_\phi(n))_n$. Let s be its limit. Now $(t_\phi(n))_n$ is also sequence of elements of E_1, thus has a converging subsequence $(t_{\phi(\psi(n))})_n$, let t be its limit, and let $\xi(n) = \phi(\psi(n))$. Then we have

- $(t_{\xi(n)})_n$ converges to t
- $(s_{\xi(n)})_n$ converges to s (since it is a subsequence of the converging sequence $(s_{\phi(n)})_n$)

We have for all $N \in \mathbb{N}$, $t \in E_N$. Indeed, let N' be such that $\xi(N') \geq N$ (such an N' exists since ξ is the extractor function and is unbounded). Then we have for all $n \geq N'$, $t_{\xi(n)} \in E_{\xi(n)} \subset E_{\xi(N')} \subset E_N$, therefore $\lim_{n\to\infty} t_{\xi(n)} \in E_N^{cl} = E_N$ since E_N is compact, hence closed. Therefore $t \in E_N$ for all $N \in \mathbb{N}$, and it follows that

$$t \in \cap_{N=1}^{\infty} E_N = E$$

Similarly, we have $s \in E$.

Now we lower bound the distance between s and t: we have

$$\begin{aligned} d(s,t) &\geq d(s_{\xi(n)}, t_{\xi(n)}) - d(s, s_{\xi(n)}) - d(t, t_{\xi(n)}) && \text{by the triangle inequality} \\ &\geq \rho - \epsilon - d(s, s_{\xi(n)}) - d(t, t_{\xi(n)}) \end{aligned}$$

and since $(t_{\xi(n)})_n$ converges to t and $(s_{\xi(n)})_n$ converges to s, there exists N such that for all $n \geq N$, $d(s, s_{\xi(n)}) \leq \epsilon$ and $d(t, t_{\xi(n)}) \leq \epsilon$. Therefore for $n \geq N$ we have

$$d(s,t) \geq \rho - 3\epsilon$$

and since $s, t \in E$, $\rho - 3\epsilon$ is a lower bound on the diameter of E. Since this is true for all $\epsilon > 0$, we obtain the result

$$d(E) \geq \rho$$

(4.4) Let X be a compact metric space and $f_n : X \to \mathbb{R}$ be continuous functions. Suppose that $\forall x \in X$ and $\forall j \in \mathbb{N}$, $f_j(x) \leq f_{j+1}(x)$. Suppose that $\forall x \in X$, $\sup_j f_j(x)$ is finite, and let $f(x) = \lim_{j\to\infty} f_j(x) = \sup_j f_j(x)$. Then if f is continuous, then (f_j) converges to f uniformly on X.

proof For all $j \in \mathbb{N}$, let

$$\begin{aligned} h_j : X &\to \mathbb{R} \\ x &\mapsto f(x) - f_j(x) \end{aligned}$$

we have

- $\forall j \in \mathbb{N}$, $\forall x \in X$, $h_j(x) \geq 0$ since $f(x) = \sup_j f_j(x) \geq f_j(x)$.
- $\forall j \in \mathbb{N}$, $\forall x \in X$, $h_{j+1}(x) \leq h_j(x)$ since $f_{j+1}(x) \geq f_j(x)$.

Now let $\alpha_j = \sup_{x \in X} h_j(x)$. Then we have $(\alpha_j)_j$ is a non-increasing sequence of non-negative reals, therefore (α_j) converges. Let α be its limit. We first show that $\alpha = 0$.

Since h_j is a continuous function (f and f_j are continuous by assumption) on the compact set X, its supremum α_j is attained. Thus there exists $x_j \in X$ such that $\alpha_j = h_j(x_j)$. This defines a sequence $(x_j)_j$ of elements of X. By compactness of X, (x_j) has a converging subsequence $(x_{n_j})_j$. Let x be its limit. Then we have for all $i > j$, $n_i > n_j$, thus $h_{n_i}(x) \leq h_{n_j}(x)$ for all $x \in X$. In particular,

$$\begin{aligned}
\alpha_{n_i} &= h_{n_i}(x_{n_i}) \\
&\leq h_{n_j}(x_{n_i}) \\
&= |f(x_{n_i}) - f_{n_j}(x_{n_i})| && \text{by definition of } h_{n_j} \\
&\leq |f(x_{n_i}) - f(x)| + |f(x) - f_{n_j}(x)| + |f_{n_j}(x) - f_{n_j}(x_{n_i})| && \text{by the triangle inequality}
\end{aligned}$$

Now fix $\epsilon > 0$. Since for fixed x, $(f_{n_j}(x))_j$ converges to $f(x)$, there exists $J \in \mathbb{N}$ such that for all $j \geq J$, $|f(x) - f_{n_j}(x)| \leq \epsilon/3$. Let us choose $j = J$. Consider the two remaining terms to bound: $|f(x_{n_i}) - f(x)|$ and $|f_{n_j}(x) - f_{n_j}(x_{n_i})|$. Since f and f_{n_i} are both continuous by assumption, both terms converge to zero when $i \to \infty$ (since $(x_{n_i})_{i>j} \to x$). Therefore $\exists I \in \mathbb{N}$ such that for all $i \geq I$, $|f(x_{n_i}) - f(x)| \leq \epsilon/3$ and $|f_{n_j}(x) - f_{n_j}(x_{n_i})| \leq \epsilon/3$. Taking $i = I$, we have

$$\begin{aligned}
\alpha_{n_i} &\leq |f(x_{n_i}) - f(x)| + |f(x) - f_{n_j}(x)| + |f_{n_j}(x) - f_{n_j}(x_{n_i})| \\
&\leq \epsilon
\end{aligned}$$

this shows that the subsequence $(\alpha_{n_i})_i$ converges to 0. But since $(\alpha_j)_j$ converges, they both have the same limit, and $\alpha = 0$.

We conclude by showing that $\alpha = 0$ implies the uniform convergence of $(f_j)_j$ to f: let $\epsilon > 0$. Since $(\alpha_j)_j$ converges to 0, there exists $J \in \mathbb{N}$ such that for all $j \geq J$, $\alpha_j \leq \epsilon$, i.e. $\sup_{x \in X} |f(x) - f_j(x))| \leq \epsilon$. Thus $(f_j)_j$ converges uniformly to f.

(4.6) There exists no function $f : \mathbb{R} \to \mathbb{R}$ that is continuous on $\mathbb{Q}$ and discontinuous on

proof We first show that the set of discontinuities of a function $f : \mathbb{R} \to \mathbb{R}$ is the countable union of closed sets: let $\text{osc}_f(x) = \lim_{\delta \to 0} \sup_{y \in B(\delta, x)} |f(y) - f(x)|$. Then we have: f is continuous at x if and only if $\text{osc}_f(x) = 0$. Therefore the set of discontinuities of f is

$$\text{disc}(f) = \cup_{n=1}^{\infty} O_{4^{-n}}$$

where

$$O_\alpha = \{x \in [0, 1] : \underset{f}{\text{osc}}(x) \geq \alpha\}$$

(Indeed, $\forall n \in \mathbb{N}$, if $x \in O_{4^{-n}}$ then $\text{osc}_f(x) \geq 4^{-n} > 0$ and x is a point of discontinuity. Conversely, if x is a point of discontinuity then $\text{osc}_f(x) > 0$, and there exists n such that $4^{-n} \leq \text{osc}_f(x)$, thus $x \in O_{4^{-n}}$)

We have

$$\cup_{n=1}^{\infty} O_{4^{-n}} = \cup_{n=0}^{\infty} O_{4^{-n}}^{cl}$$

(since $\forall n \geq 0$, $O_{4^{-n}}^{cl} \subseteq O_{4^{-(n+1)}}$ by Lemma 2.55 from the book, we have the inclusion $\supseteq$, and since $\forall n \geq 1$, $O_{4^{-n}} \subseteq O_{4^{-n}}^{cl}$, we have the inclusion $\subseteq$)

Therefore

$$\text{disc}(f) = \cup_{n=0}^{\infty} O_{4^{-n}}^{cl}$$

and the set of discontinuities of f is the countable union of closed subsets.

Now assume by contradiction that such a function exists: $f : \mathbb{R} \to \mathbb{R}$ continuous on $\mathbb{Q}$ and discontinuous on $\mathbb{R} \setminus \mathbb{Q}$. Then the set of discontinuities of f, $\mathbb{R} \setminus \mathbb{Q}$, is the countable union of closed subsets. Let $\mathbb{R} \setminus \mathbb{Q} = \cup_{n=1}^{\infty} C_n$ where C_n is closed. But since $\mathbb{R} \setminus \mathbb{Q}$ contains no open interval, each C_n contains no open interval. Therefore each C_n has open interior, and since it is closed, it is nowhere dense. Now we can write $\mathbb{R}$ as

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}) = \cup_{q \in \mathbb{Q}} \{q\} \cup \cup_{n \in \mathbb{N}} C_n$$

therefore $\mathbb{R}$ is a countable union of closed nowhere dense sets (for every $q \in \mathbb{Q}$, the singleton $\{q\}$ is closed and nowhere dense). This contradicts Baire's theorem. Therefore, no such function f exists.

(4.7) There exists a continuous function $f : (0,1) \to \mathbb{R}$ which is nowhere differentiable ($f'(x)$ exists for no x).

proof Consider the metric space $X = C([0,1], \mathbb{R})$ of all continuous functions $f : [0,1] \to \mathbb{R}$, with the metric

$$\begin{aligned} d : X \times X &\to \mathbb{R} \\ (f,g) &\mapsto d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)| \end{aligned}$$

(X, d) is a complete metric space. For $N \in \mathbb{N}$ let $\mathcal{F}_N = \{f \in X | \exists y \in [0,1] : \forall x \in [0,1] |f(x) - f(y)| \leq N|x-y|\}$, be the subset of functions that are N-Lipschitz at some point y. Then

(i) $\mathcal{F}_N$ is a closed subset of X. Indeed, let $(f_n)_n$ be a converging sequence of elements of $\mathcal{F}_N$, and let f be its (uniform) limit. In particular, f is continuous.

We have $\forall n \in \mathbb{N}$, there exists $y_n \in [0,1]$ such that $\forall x \in [0,1]$, $|f(x) - f(y_n)| \leq N|x - y_n|$. Since $y_n \in [0,1] \forall n$ and $[0,1]$ is compact, (y_n) has a converging subsequence $(y_{n_k})_k$. Let y be the limit of $(y_{n_k})_k$. Then we have $\forall x \in [0,1]$, by the triangle inequality

$$|f(x) - f(y)| \leq |f(x) - f_{n_k}(x)| + |f_{n_k}(x) - f_{n_k}(y_{n_k})| + |f_{n_k}(y_{n_k}) - f(y_{n_k})| + |f(y_{n_k}) - f(y)| \quad (1)$$

Then we have by definition of y_{n_k},

$$\begin{aligned} |f_{n_k}(x) - f_{n_k}(y_{n_k})| &\leq N|x - y_{n_k}| \\ &\leq N|x-y| + N|y - y_{n_k}| \end{aligned} \quad (2)$$

by the triangle inequality. Now fix $\epsilon > 0$. Since f is continuous (uniform limit of continuous functions on compact sets), there exists $\eta > 0$ such that for all y' such that $|y - y'| \leq \eta$, $|f(y) - f(y')| \leq \epsilon$.

And since $(y_{n_k}) \to y$, there exists K_1 such that $\forall k \geq K_1$, $|y_{n_k} - y| \leq \min(\epsilon/N, \eta)$. And since (f_n) converges uniformly to f, there exists N_2 such that for all $n \geq N_2$, $\sup_{x\in[0,1]} |f_n(x) - f(x)| \leq \epsilon$. Let $K_2 = \min\{k : n_k \geq N_2\}$. Then we have for all $k \geq \max(K_1, K_2)$,

- $\sup_{x\in[0,1]} |f_{n_k}(x) - f(x)| \leq \epsilon$, and in particular, $|f_{n_k}(y_{n_k}) - f(y_{n_k})| \leq \epsilon$ and $|f_{n_k}(x) - f(x)| \leq \epsilon$.
- $|y - y_{n_k}| \leq \epsilon/N$ thus $|f_{n_k}(x) - f_{n_k}(y_{n_k})| \leq N|x-y| + \epsilon$ by (2)
- $|y - y_{n_k}| \leq \eta$ thus $|f(y_{n_k}) - f(y)| \leq \epsilon$

using these inequalities in (1), we have

$$|f(x) - f(y)| \leq N|x-y| + 4\epsilon$$

Since this is true for all $\epsilon > 0$, we conclude that

$$|f(x) - f(y)| \leq N|x-y|$$

for all x. Therefore $f \in \mathcal{F}_N$, which proves that F_N is closed

(ii) $\mathcal{F}_N$ has empty interior. Indeed, let $f \in \mathcal{F}_N$, and let $\epsilon > 0$. We show that the open ball $B(3\epsilon, f)$ contains at least one element that is not in $\mathcal{F}_N$. First, we approximate the function f with a piecewise affine function $\tilde{f}$ such that $d(f, \tilde{f}) \leq \epsilon$. Then we show that we can construct a piecewise affine function g such that g is not in $\mathcal{F}_N$, and $d(g, \tilde{f}) \leq \epsilon$.

- constructing $\tilde{f}$: since f is continuous on the compact $[0,1]$, it is uniformly continuous. Thus there exists $\delta > 0$ such that for all $x, x' \in [0,1]$ such that $|x - x'| \leq \delta$, $|f(x) - f(x')| \leq \epsilon/2$. Consider the finite collection $\{a_i\}_{i\in\{0,\ldots k\}}$ such that $a_i = i\delta$ for $i < k$, where $k = \min\{n : n\delta > 1\}$ and $a_k = 1$. Then define $\tilde{f}$ to coincide with f on the collection $\{a_i\}$, and to be affine on each interval $[a_i, a_{i+1}]$. Then we have for all $x \in [a_i, a_{i+1}]$, $|f(x) - \tilde{f}(x)| \leq \epsilon$, therefore

$$d(f, \tilde{f}) = \sup_{x\in[0,1]} |f(x) - \tilde{f}(x)| \leq \epsilon$$

- constructing g: since $\tilde{f}$ is piecewise affine, it is Lipschitz continuous, and there exists M such that for all $x, x' \in [0,1]$, $|\tilde{f}(x) - \tilde{f}(x')| \leq M|x - x'|$. Now define $\tilde{g}$ to be the piecewise affine function defined as follows
  - Consider the finite collection $\{b_i\}_{i \in \{0,\ldots,k'\}}$ where $k' = \min\{n : n\epsilon' > 1\}$, $\epsilon' = \epsilon/(M + N + 1)$, $b_i = i\epsilon'$ for $i < k'$, and $b_{k'} = 1$.
  - $\tilde{g}$ is affine on each interval $[b_i, b_{i+1}]$, and

$$\tilde{g}(b_i) = \begin{cases} -\epsilon/2 & \text{if } i \text{ is even} \\ \epsilon/2 & \text{if } i \text{ is odd} \end{cases}$$

  Then we have $\forall x, x' \in [0,1]$, $|\tilde{g}(x) - \tilde{g}(x')| \geq (M + N + 1)|x - x'| > (M + N)|x - x'|$.

  Now let $g = \tilde{f} + \tilde{g}$. Then we have $g \notin \mathcal{F}_N$ since $\forall x, x' \in [0,1]$,

$$\begin{aligned} |g(x) - g(x')| &= |\tilde{g}(x) + \tilde{f}(x) - \tilde{g}(x') - \tilde{f}(x')| \\ &\geq |\tilde{g}(x) - \tilde{g}(x')| - |\tilde{f}(x) - \tilde{f}(x')| \\ &> (M + N)|x - x'| - M|x - x'| \\ &= N|x - x'| \end{aligned}$$

Therefore we have $g \notin \mathcal{F}_N$, and $g \in B(3\epsilon, f)$ since

$$d(g, f) \leq d(f, \tilde{f}) + d(\tilde{f}, g) \leq 2\epsilon < 3\epsilon$$

which proves that $F\mathcal{F}_N$ has empty interior.

(iii) By the Baire Category theorem, a complete subset is not of the first category, i.e. is not the countable union of closed nowhere dense subsets. Therefore,

$$X \neq \cup_{N=1}^{\infty} \mathcal{F}_N$$

(iv) Let $f \in X \setminus \cup_{N=1}^{\infty} \mathcal{F}_N$. Then f is continuous (by definition of X) and nowhere differentiable. To prove this, we show that if the derivative exists at some point, then the function is in $\mathcal{F}_n$ for some N. Indeed, let g be a continuous function, and assume that there exists y such that $g'(y)$ exists, i.e. the function

$$\begin{aligned} \tau_y : [0,1] \setminus \{y\} &\to \mathbb{R} \\ x &\mapsto \frac{g(y) - g(x)}{y - x} \end{aligned}$$

has a limit when $x \to y$, and its limit is $g'(y)$. Fix $\epsilon > 0$. Since τ_y is continuous, $\exists \eta > 0$ such that for all $x \in [0,1] \setminus \{y\}$ such that $|x - y| \leq \eta$, $|\tau_y(x) - g'(y)| \leq \epsilon$ i.e.

$$|g(y) - g(x)| \leq \epsilon|y - x| \forall x \text{ such that } |y - x| \leq \eta$$

Since $x \mapsto |g(y) - g(x)|$ is a continuous function on the compact $[0,1]$, it is bounded, and we have $|g(y) - g(x)| \leq M$ for some M. Now consider x such that $|y - x| \geq \eta$. We have $\frac{|g(y)-g(x)|}{|x-y|} \leq \frac{M}{\eta}$, i.e.

$$|g(y) - g(x)| \leq \frac{M}{\eta}|x - y| \forall x \text{ such that } |y - x| \geq \eta$$

Therefore

$$|g(y) - g(x)| \leq \max\left(\frac{M}{\eta}, \epsilon\right)|x - y| \forall x \in [0,1]$$

and $g \in \mathcal{F}_N$ where $N = \max\left(\frac{M}{\eta}, \epsilon\right)$.