(3.1) Consider a metric space \((X, d)\).

(a) Let \((x_n)\) be a Cauchy sequence, and assume that a subsequence \((x_{\phi(n)})\) converges to \(l \in X\). Then \((x_n)\) also converges to \(l\).

\[\text{proof}\] Let \(\epsilon > 0\). Since \(x_n\) is a Cauchy sequence, then \(\exists N_1 \in \mathbb{N}\) such that \(\forall n, m \geq N_1, d(x_n, x_m) \leq \epsilon/2\).

And since \((x_{\phi(n)})\) converges to \(l\), then \(\exists N_2 \in \mathbb{N}\) such that \(\forall n > N_2, d(x_{\phi(n)}, l) \leq \epsilon/2\). Let \(N = \max(N_1, N_2)\).

Then we have \(\forall n \geq N, d(x_n, l) \leq d(x_n, x_{\phi(n)}) + d(x_{\phi(n)}, l) \leq \epsilon/2 + \epsilon/2\) (since \(\phi(n) \geq n\)). Thus \((x_n)\) converges to \(l\).

(b) Let \((x_n)\) be a Cauchy sequence. Then \((x_n)\) is bounded.

\[\text{proof}\] Fix \(\epsilon > 0\). We have \(\exists N \in \mathbb{N}\) such that \(\forall n, m \geq N, d(x_n, x_m) \leq \epsilon\). In particular, we have \(\forall n \geq N, d(x_n, x_N) \leq \epsilon\). Let \(r = \max(\epsilon, d(x_0, x_N), \ldots, d(x_{N-1}, x_N))\). Then we have \(\forall n \in \mathbb{N}, d(x_n, x_N) \leq r\), thus \((x_n)\) is bounded.

(3.2) Let \((X, d)\) be a metric space, and \(Y \subset X\). Let \(d'\) be the restriction of \(d\) to \(Y\). Then if \((Y, d')\) is complete, then \(Y\) is a closed subset of \(X\).

\[\text{proof}\] Let \((x_n)\) be a converging sequence of \((X, d)\), such that \(\forall n, x_n \in Y\), and let \(l\) be its limit. To show that \(Y\) is closed, it suffices to show that \(l \in Y\) for any such sequence. First, \((x_n)\) is a Cauchy sequence of \((Y, d')\): since \((x_n)\) converges in \((X, d)\), \(\forall \epsilon > 0, \exists N\) such that \(\forall n \geq N, d(x_n, l) \leq \epsilon/2\), thus \(\forall n, m \geq N, d'(x_n, x_m) = d(x_n, x_m) \leq d(x_n, l) + d(l, x_m) \leq \epsilon\).

Since \((Y, d')\) is complete, the Cauchy sequence \((x_n)\) converges and its limit is in \(Y\). Let \(l' \in Y\) be the limit of \((x_n)\) as a converging sequence of \((Y, d')\). Then \(l'\) is also a limit of \((x_n)\) as a converging sequence of \((X, d)\) since \(\forall \epsilon > 0, \exists N\) such that \(d'(x_n, l') \leq \epsilon\), thus \(d(x_n, l') \leq \epsilon\). By uniqueness of the limit, we have \(l = l'\), thus \(l \in Y\).

(3.3) Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces, \(f : X \to Y\) a continuous function, and \(G = \{(x, y) \in X \times Y : y = f(x)\}\). Then \(G\) is closed subset of \((X \times Y, d)\) where \(d\) is the product metric \(d(p_1, p_2) = \max(d_X(x_1, x_2), d_Y(y_1, y_2))\).

\[\text{proof}\] \(G\) is the inverse image of the closed set \(\{0\}\) by the continuous function

\[h : (X, Y) \to \mathbb{R}\]

\[(x, y) \mapsto d(f(x), y)\]

Indeed, \((x, y) \in G\) if and only if \(f(x) = y\), if and only if \(d(f(x), y) = 0\). \(h\) is continuous as the composition of the continuous functions \(h_1\) and \(h_2\) given by

\[X \times Y \xrightarrow{h_1} Y \times Y \xrightarrow{h_2} \mathbb{R}\]

where

\[h_1 : X \times Y \to Y \times Y\]

\[(x, y) \mapsto (f(x), y)\]
h_2 : Y \times Y \rightarrow \mathbb{R} \\
(y, y') \mapsto d(y, y')

(3.4)

(a) Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces, and let \(f : X \rightarrow Y\) be a homeomorphism. Then \((x_n)\) converges in \(X\) if and only if \((f(x_n))\) converges in \(Y\). Further, \(x_n \rightarrow x\) if and only if \(f(x_n) \rightarrow f(x)\).

**Proof** Only if: let \((x_n)\) be a converging sequence. Let \(x\) be its limit. Let \(\epsilon > 0\). Since \(f\) is continuous, \(\exists \eta > 0\) such that if \(d_X(x_n, x) \leq \eta\), then \(d_Y(f(x_n), f(x)) \leq \epsilon\). And since \((x_n)\) converges to \(x\), \(\exists N \in \mathbb{N}\) such that if \(n \geq N\), then \(d_X(x_n, x) \leq \eta\). Therefore \(\forall n \geq N\), \(d_Y(f(x_n), f(x)) \leq \epsilon\). This proves that \((f(x_n))\) converges, and that its limit is \(f(x)\).

If: Assume \((f(x_n))\) converges. Let \(y\) be its limit. Let \(\epsilon > 0\). Since \(f^{-1}\) is continuous, there exists \(\eta > 0\) such that if \(d_Y(f(x_n), y) \leq \eta\), then \(d_X(f^{-1}(f(x_n)), f^{-1}(y)) \leq \epsilon\). And since \((f(x_n))\) converges to \(y\), \(\exists N \in \mathbb{N}\) such that if \(n \geq N\), then \(d_Y(f(x_n), y) \leq \eta\). Thus \(\forall n \geq N\), \(d_X(f^{-1}(f(x_n)), f^{-1}(y)) \leq \epsilon\), thus \((x_n)\) converges, and its limit is \(f^{-1}(y)\). Further, if \(f(x_n) \rightarrow f(x)\), then \((x_n) \rightarrow f^{-1}(f(x)) = x\).

(b) \(\mathbb{R}\) is not homeomorphic to \(\mathbb{Q}\).

**Proof** since \(\mathbb{R}\) is not countable, there are no injective maps from \(\mathbb{R}\) to \(\mathbb{Q}\). Therefore, there are no bijective maps from \(\mathbb{R}\) to \(\mathbb{Q}\).

**Proof** without using \(\mathbb{R}\) uncountable: assume by contradiction that \(f : \mathbb{Q} \rightarrow \mathbb{R}\) is a homeomorphism. Let \((x_n)\) be a sequence of rationals that converge to an irrational \(x\) ((\(x_n\)) does not converge in \(\mathbb{Q}\), but converges in \(\mathbb{R}\)). Then \((f(x_n))\) is a converging sequence:

- \((x_n)\) is a Cauchy sequence in \(\mathbb{Q}\): let \(d\) be the usual metric on \(\mathbb{R}\) and \(d'\) be its restriction on \(\mathbb{Q}\). Then \(\forall \epsilon > 0\), \(\exists N\) such that \(\forall n \geq N\), \(d(x, x_n) \leq \epsilon/2\). Then \(\forall m, n \geq N\), \(d'(x_n, y) = d(x, x_n) \leq d(x_n, x) + d(x, x_m) \leq \epsilon\).

- \((f(x_n))\) is a Cauchy sequence in \(\mathbb{R}\) as the image bu a continuous function of a Cauchy sequence. Since \(f\) is continuous, \(\forall \epsilon > 0\), there exists \(\eta > 0\) such that if \(d(x, x') \leq \eta\) then \(d(f(x), f(x')) \leq \epsilon\). Then \((x_n)\) is Cauchy, there exists \(N \in \mathbb{N}\) such that for all \(n, m \geq N\), \(d(x_n, x_m) \leq \eta\). Thus for all \(m, n \geq N\), \(d(f(x_n), f(x_m)) \leq \epsilon\).

- \(\mathbb{R}\) is complete. Thus \((f(x_n))\) converges.

Let \(y\) be the limit of \((f(x_n))\). But from (a), \((f(x_n))\) converges to \(y\) only if \((x_n)\) converges to \(f^{-1}(y)\). But \((x_n)\) does not converge in \(\mathbb{Q}\), contradiction.

(c) There is no bounded continuous injective mapping \(f : \mathbb{R} \rightarrow \mathbb{R}^2\) such that the range \(f(\mathbb{R})\) is a closed subset of \(\mathbb{R}^2\).

**Proof** Assume such a function exists, and let \(B = f(\mathbb{R})\). \(B\) is a closed bounded subset of \(\mathbb{R}^2\), thus is compact. Then \(\hat{f}\) defined by

\[
\hat{f} : \mathbb{R} \rightarrow B \\
x \mapsto f(x)
\]

is a homeomorphism. Now consider the sequence \((x_n)\) defined by \(x_n = n\). Then \((f(x_n))\) is a sequence of elements of the compact set \(B\), therefore admits a converging subsequence \((f(x_{\phi(n)}))_n\). Let \(y\) be its limit. Then by (a), \((x_{\phi(n)})\) must converge, and its limit must be \(f^{-1}(y)\). However, \((x_{\phi(n)})\) cannot converge since it is unbounded. This leads to a contradiction and proves the result.
(3.5) Let \( E \subset \mathbb{R}^n \) be uncountable. Then there exists \( x \in \mathbb{R}^n \) such that for any open ball \( B(x, r) \), \( B(x, r) \cap E \) is uncountable.

Proof by contrapositive: assume that for all \( x \in \mathbb{R}^n \), there exists \( r_x > 0 \) such that \( B(r_x, x) \cap E \) is countable. Let \( B_x = B(r_x, x) \cap E \).

Now consider the metric space \( (E, d') \), where \( d' \) is the restriction of the metric \( d \) to \( E \). Then the open sets of \( (E, d') \) are the intersections of the open sets of \( (\mathbb{R}^n, d) \) with \( E \). In particular, \( B_x = B(r_x, x) \cap E \) is open in \( (E, d') \) for every \( x \). Therefore \( \{B_x\}_{x \in E} \) is an open cover of \( E \) (by definition of \( B_x \)), we have \( B_x \subseteq E \), therefore \( \bigcup_{x \in E} B_x \subseteq E \). Conversely, for all \( x_0 \in E, x_0 \in B_{x_0} \), thus \( x_0 \in \bigcup_{x \in E} B_x \), therefore \( E \subseteq \bigcup_{x \in E} B_x \).

Now since \( E \) is separable (\( E \cap \mathbb{Q}^n \) is countable and dense in \( E \)), every open cover has a countable subcover. Let \( \{B_{x_n}\}_{n \in \mathbb{N}} \) be such a subcover. Then we have \( E = \bigcup_{n \in \mathbb{N}} B_{x_n} \), thus \( E \) is a countable union of countable sets (by assumption, \( B_x \) is countable for every \( x \)), therefore \( E \) is countable (from problem set 1, a countable union of countable sets is countable).

(3.6) Let \( f \) be the uniform limit of the functions \( f_n : [0, 1] \to [0, 1]^2 \) defined inductively by the Hilbert construction.

Then we have \( \forall x \in [0, 1] \)

\[
\|f_n(x) - f_{n+m}(x)\|_2 \leq \sum_{k=0}^{m-1} \|f_{n+k}(x) - f_{n+k+1}(x)\|_2
\]

and the distance between \( \|f_{n+k}(x) - f_{n+k+1}(x)\|_2 \) is at most the diagonal of the square at the \( n + k \)-th step, i.e. \( \|f_{n+k}(x) - f_{n+k+1}(x)\|_2 \leq \sqrt{2}(1/2)^{n+k} \). Thus

\[
\|f_n(x) - f_{n+m}(x)\|_2 \leq \sqrt{2}(1/2)^n \sum_{k=0}^{m-1} (1/2)^k \leq \sqrt{2}(1/2)^n (1/2)
\]

thus

\[
\|f_n - f_{n+m}\|_\infty \leq \sqrt{2}(1/2)^n (1/2)
\]

Therefore \( (f_n) \) is a Cauchy sequence in the complete subspace of continuous functions on \( [0, 1]^2 \) (with the metric induced by the infinite norm), thus converges. Let \( f \) be its limit. Then \( f \) is continuous (as a uniform limit of continuous functions) and is surjective: let \( y \in [0, 1]^2 \), and fix \( \epsilon > 0 \). Then we can find \( x \) such that \( \|f(x) - y\| \leq \epsilon \): there exists \( N_1 \in \mathbb{N} \) and \( x_1 \in [0, 1] \) such that for all \( n \geq N_1 \), \( \|f_n(x) - y\| \leq \epsilon/2 \). And since \( (f_n) \) converges uniformly to \( f \), there exists \( N_2 \) such that for all \( n \geq N_2 \), \( \|f - f_n\|_\infty \leq \epsilon/2 \). Let \( N = \max(N_1, N_2) \), then \( \|f(x) - y\| \leq \|f - f_N\|_\infty + \|f_N(x) - y\| \leq \epsilon \).

Now construct a sequence \( (x_n) \) of elements of \( [0, 1] \) such that \( \forall n \|f(x_n) - y\| \leq 1/n \). Since \([0, 1]\) is compact, \( (x_n) \) admits a converging subsequence \( (x_{n+}\)\( ) \), let \( x \) be its limit. Then \( f(x_{n+}) \) is converging by continuity of \( f \), and its limit is \( f(x) \) by continuity, and \( y \) by construction, therefore \( f(x) = y \) and \( f \) is surjective.

(3.7) Let \( O \) be a nonempty open subset of \( \mathbb{R} \). Then there exists a countable collection \( \{I_j\} \) of pairwise disjoint open intervals such that \( O = \bigcup_j I_j \).

Proof: For every \( x \in O \) let \( C_x = \{J \subset O : J \) is an open interval containing \( x\} \), and \( I_x = \bigcup_{J \in C_x} J \). \( I_x \) is by construction an open interval, and a subset of \( O \) that contains \( x \).

We have \( \{I_x\}_{x \in O} \) is an open cover of \( O \) since

- \( \forall x \in O, \forall J \in C_x, J \subset O \), thus \( \forall J \in C_x \), \( J \subset O \), i.e. \( I_x \subseteq O \). Thus \( \bigcup_{x \in O} I_x \subseteq O \).

- \( \forall x \in O \), let \( x \in I_x \), thus \( x \in \bigcup_{x \in O} I_x \). Therefore \( O \subseteq \bigcup_{x \in O} I_x \). This proves that \( O = \bigcup_{x \in O} I_x \).

The distinct elements of \( \{I_x\}_{x \in O} \) are disjoint: to prove this (by contrapositive), assume that \( I_x \cap I_{x'} \neq \emptyset \). Then \( I_x \cup I_{x'} \) is an open interval, is a subset of \( O \), and contains \( x \), thus it is a member of \( C_x \), thus \( I_x \cap I_{x'} \subseteq I_x \).
Obviously, we also have \( I_x \subseteq I_x \cap I_{x'} \), therefore we have \( I_x = I_x \cap I_{x'} \). Similarly, we have \( I_{x'} = I_x \cap I_{x'} \), which proves the result.

Finally, \( \{I_x\}_{x \in \mathcal{O}} \) is countable, since every distinct element \( I_x \) contains a rational, say \( r(I_x) \in I_x \) (since \( \mathbb{Q} \) is dense in \( \mathbb{R} \)), one can construct a map (using the axiom of choice)

\[
i : \{I_x\}_{x \in \mathcal{O}} \to \mathbb{Q} \\
i(I_x) \mapsto r(I_x)
\]

\( i \) is injective (since the elements of \( \{I_x\}_{x \in \mathcal{O}} \) are disjoint, and \( r(I_x) \in I_x \)), and \( \mathbb{Q} \) is countable, therefore \( \{I_x\}_{x \in \mathcal{O}} \) is countable.