Consider the metric space $\mathbb{Q}$ with the metric $d(x,y) = |x-y|$. Define the set $X = \{x \in \mathbb{Q} \mid |x| < \sqrt{2}\}$ is both open and closed.

Let $(X,d)$ be $\mathbb{R}^2$ with the standard metric $d(a,b) = \|a-b\|_2 = ((a_1 - b_1)^2 + (a_2 - b_2)^2)^{1/2}$. Let $A = \{(x,y) \in \mathbb{Q}^2 : x^2 + y^2 < 1\} = \{a \in \mathbb{Q}^2 : \|a\|_2 < 1\}$ and $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} = \{b \in \mathbb{R}^2 : \|b\|_2 \leq 1\}$. Then $\overline{bd}A = B$.

proof: We use the fact that $\forall b \in B$, $\forall \epsilon > 0$,
(a) $B(\epsilon, b) \cap (A - \{b\}) \neq \emptyset$
(b) $B(\epsilon, b) \cap (A^c - \{b\}) \neq \emptyset$

To prove (a), let $b \in B$, and fix $\epsilon > 0$. Let $b' = (1 - \epsilon/2)b$ (this is to handle the case where $b$ lies on the sphere, see figure 1). Then we have $\|b'\|_2 = (1 - \epsilon/2)\|b\|_2 < 1$. Let $\epsilon' = \frac{\epsilon}{2 \sqrt{2}} > 0$. Then we have $(b' - \epsilon', b' + \epsilon')$ contains elements of $\mathbb{Q}$. Similarly, $(b'_2 - \epsilon', b'_2 + \epsilon')$ contains elements of $\mathbb{Q}$. Therefore there exists $c \in \mathbb{Q}^2$ such that $c \neq b'$ such that $|c_1 - b'_1| < \epsilon'$ and $|c_2 - b'_2| < \epsilon'$, and $c$ satisfies $\|c - b'\|_2 = \sqrt{(c_1 - b'_1)^2 + (c_2 - b'_2)^2} < \sqrt{2\epsilon'^2} = \epsilon/2$. And we have using the triangle inequality

$$\|c - b'\|_2 \leq \|c - b\|_2 + \|b' - b\|_2$$

$$= \|c - b\|_2 + \frac{\epsilon}{2} \|b\|_2$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \|b\|_2$$

$$\leq \epsilon$$
and

\[ \|c\|_2 \leq \|b'\|_2 + \|c - b'\|_2 \leq (1 - \frac{\epsilon}{2})\|b\|_2 + \frac{\epsilon}{2} \]

\[ = \|b\|_2 + \frac{\epsilon}{2}(1 - \|b\|_2) \leq \|b\|_2 \leq 1 \]

Therefore \( c \in B(\epsilon, b) \), \( c \in A \) and \( c \neq b \), i.e. that \( c \in B(\epsilon, b) \cap (A - \{b\}) \), which proves (a). To prove (b), a similar argument is used, where \( c \) is now taken to be in \((\mathbb{R} - \mathbb{Q})^2\).

![Figure 1: Illustration of the proof of (a) and (b). Left: case where \( b \) lies on the unit sphere \( \{a \in \mathbb{R}^2 : \|a\|_2 = 1\} \), right: generic case.](image)

**Proof of** \( B = \text{bd} A \): To prove that \( B \subseteq \text{bd} A \), let \( b \in B \), and construct a sequence \((a_n)\) as follows: \( \forall n \in \mathbb{N} \), let \( \epsilon = \frac{1}{n+1} \), and define \( a_n \) to be any element in \( B(\epsilon, b) \cap (A - \{b\}) \) (nonempty by (a)). \((a_n)\) thus defined is a sequence of \( A - \{b\} \) that converges to \( b \), thus \( b \) is a limit point of \( A \). Now define sequence \((b_n)\) as follows: \( \forall n \in \mathbb{N} \), let \( \epsilon = \frac{1}{n+1} \), and define \( b_n \) to be any element in \( B(\epsilon, b) \cap (A^c - \{b\}) \) (nonempty by (b)). \((b_n)\) thus defined is a sequence of \( A^c - \{b\} \) that converges to \( b \), thus \( b \) is a limit point of \( A^c \). This proves that \( B \subseteq \text{bd} A \).

Conversely, if \( b \notin B \), then \( \|b\|_2 > 1 \), and there exists \( \epsilon > 0 \) such that \( B(\epsilon, b) \cap A = \emptyset \) (for example take \( \epsilon = 1 - \|b\|_2 \)). Thus \( b \) is not a limit point of \( A \), and as a consequence \( b \notin \text{bd} A \).

(2.5) Let \( A, B \) be subsets of a metric space. Then \( A^o \cap B^o = (A \cap B)^o \).

**Proof:** \( \subseteq \): Let \( x \in A^o \cap B^o \). Since \( x \in A^o \), \( \exists \epsilon_1 > 0 \) such that \( B(\epsilon_1, x) \subseteq A \), and since \( x \in B^o \), \( \exists \epsilon_2 > 0 \) such that \( B(\epsilon_2, x) \subseteq B \). Let \( \epsilon = \min(\epsilon_1, \epsilon_2) \). Then we have \( B(\epsilon, x) \subseteq A \cap B \), thus \( x \in (A \cap B)^o \).

\( \supseteq \): Let \( x \in (A \cap B)^o \). Then exists \( \epsilon > 0 \) such that \( B(\epsilon, x) \subseteq (A \cap B) \). Thus we have \( B(\epsilon, x) \subseteq A \) and \( B(\epsilon, x) \subseteq B \), thus \( x \in A^o \) and \( x \in B^o \), therefore \( x \in A^o \cap B^o \).

(2.6) Let \( A = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, \text{ and } xy > 1\} \). Then \( A \) is an open set.

**Proof:** Consider the metric

\[ d : \mathbb{R}^2 \to \mathbb{R} \\
(a, b) \mapsto \|a - b\|_1 = |a_1 - b_1| + |a_2 - b_2| \]

Let \( f : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto x \), \( g : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto x_2 \), and \( h : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto x_1x_2 \). The function \( f \) is continuous since \( \forall a, b \in \mathbb{R}^2 \), we have

\[ d(f(a), f(b)) = d(a_1, b_1) = |a_1 - b_1| \leq |a_1 - b_1| + |a_2 - b_2| = d(a, b) \]
therefore $\forall \epsilon > 0, d(a, b) \leq \epsilon \Rightarrow d(f(a), f(b)) \leq \epsilon$ and $f$ is continuous. Similarly, $g$ is continuous. Finally, $h$ is continuous as the product of continuous functions $f$ and $g^1$.

Then we have $A = f^{-1}((0, +\infty)) \cap g^{-1}((0, +\infty)) \cap h^{-1}(1, +\infty)$ is the intersection of open sets (inverse image by continuous functions of open sets) and is therefore open.

(2.7) Consider $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 
0 & \text{if } x = 0 \\
1/x & \text{if } x > 0
\end{cases}$$

The graph of $f$, $G = \{(x, 1/x), x \in (0, 1]\} \cup (0, 0)$ is closed, but $f$ is not continuous.

(2.8) Let $(x_n)$ be a sequence of points in a metric space $(X, d)$, and let $z \in X$. Suppose that any subsequence of $(x_n)$ has a subsequence that converges to $z$. Then $\lim_{n \to \infty} x_n = z$. \textit{proof:} by contrapositive. Assume $(x_n)$ does not converge to $z$. Then $\exists \epsilon > 0$ such that

$$\forall N \in \mathbb{N}, \exists n \geq N \text{ such that } d(x_n, z) > \epsilon \quad (1)$$

We use this fact to construct a subsequence $(x_{\phi(n)})_n$ ($\phi : \mathbb{N} \to \mathbb{N}$ strictly increasing) such that $\forall n \in \mathbb{N}$, $d(x_{\phi(n)} - z)$ $\epsilon$. The subsequence $(x_{\phi(n)})$ is defined by induction:

- We know by (1) (taking $N = 0$) that $\exists n_0 \geq 0$ such that $d(x_{n_0}, z) > \epsilon$. Define $\phi(0) = n_0$
- Assume $(\phi(n))$ defined. Then we know by (1) (taking $N = \phi(n) + 1$) that $\exists n_1 \geq \phi(n) + 1$ such that $d(x_{n_1}, z)$ $\epsilon$. Define $\phi(n + 1) = n_1$. We have $\phi(n + 1) > \phi(n)$.

By construction, the subsequence $(x_{\phi(n)})_n$ satisfies $\forall n \in \mathbb{N}, d(x_{\phi(n)}, z) > \epsilon$, therefore does not have a subsequence that converges to $z$. This proves the contrapositive of the desired result.

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1\text{proof that the product of continuous functions is continuous: let } f : X \to \mathbb{R} \text{ and } g : X \to \mathbb{R} \text{ continuous at } x_0, \text{ where } (X, d) \text{ is a metric space. Let } \epsilon > 0. \text{ Since } f \text{ and } g \text{ are continuous, } \forall \epsilon' > 0, \exists \delta_1 > 0, \delta_2 > 0 \text{ such that } d(x, x_0) \leq \delta \Rightarrow |f(x) - f(x_0)| \leq \epsilon', \text{ and } d(x, x_0) \leq \delta \Rightarrow |g(x) - g(x_0)| \leq \epsilon' \quad (f(x)g(x) - f(x_0)g(x_0)) = |(f(x) - f(x_0))(g(x) - g(x_0)) - f(x_0)g(x_0)| \\
= |f(x) - f(x_0)||g(x) - g(x_0)| + |f(x) - f(x_0)||g(x_0) + f(x_0)(g(x) - g(x_0))| \\
\leq |f(x) - f(x_0)||g(x) - g(x_0)| + |f(x) - f(x_0)||g(x_0)| + |f(x_0)||g(x) - g(x_0)| \\
\text{thus if } d(x, x_0) \leq \min(\delta_1, \delta_2), \quad |f(x)g(x) - f(x_0)g(x_0)| \leq \epsilon'^2 + \epsilon'(|g(x_0)| + |f(x_0)|) \\
\text{taking } \epsilon' \text{ small enough, we can ensure } |f(x)g(x) - f(x_0)g(x_0)| \leq \epsilon$
(2.9) Let $A$ be a subset of a metric space $(X,d)$, and let

$$S = \{A, A^o, A^{cl}, (A^o)^o, (A^{cl})^o, ((A^o)^o)^o, ((A^{cl})^o)^o\}$$

We show that $S$ is invariant by the following operations: interior, closure and complement. We first show some facts:

$$(A^o)^o = (A^{cl})^o \quad \text{(1)}$$

$$(A^{cl})^o = (A^o)^o \quad \text{(2)}$$

$$(A^o)^{cl} = ((A^{cl})^o)^o \quad \text{(3)}$$

$$((A^o)^{cl})^o = (A^{cl})^o \quad \text{(4)}$$

$$((A^{cl})^o)^o = (A^o)^{cl} \quad \text{(5)}$$

$$(((A^o)^{cl})^o)^o = ((A^{cl})^o)^{cl} \quad \text{(6)}$$

$$(((A^{cl})^o)^o)^o = ((A^o)^{cl})^o \quad \text{(7)}$$

$$(((A^o)^{cl})^o)^o = ((A^{cl})^o)^{cl} \quad \text{(8)}$$

**proof:**

(1) we have

$$x \in (A^o)^o \iff \forall \epsilon > 0, B(\epsilon, x) \not\subseteq A$$

$$\iff \forall \epsilon > 0, B(\epsilon, x) \cap A^c \neq \emptyset$$

$$\iff x \in (A^{cl})^o$$

(5) We have $A^o \subseteq (A^{cl})^o$ by definition of the closure. Thus by monotonicity of the interior, $(A^o)^o \subseteq ((A^{cl})^o)^o$, i.e. $A^o \subseteq ((A^{cl})^o)^o$, and by monotonicity of the closure, $(A^o)^{cl} \subseteq (((A^{cl})^o)^o)^{cl}$.

To prove the reverse inclusion, we have by definition of the interior, $((A^{cl})^o)^o \subseteq (A^o)^{cl}$, thus taking the closure, $(((A^{cl})^o)^o)^{cl} \subseteq ((A^o)^{cl})^o = (A^{cl})^o$.

(2) replacing $A$ with $A^c$ in (1), we have $((A^c)^o)^c = A^{cl}$, thus taking the complement, we obtain (2)

(7) replacing $A$ with $A^c$ in (5), we have

$$(((A^c)^o)^{cl})^o = ((A^{cl})^o)^{cl}$$

and applying identities (1) and (2) repeatedly, we have

$$(((((A^c)^o)^{cl})^o)^{cl})^o = (((A^{cl})^o)^{cl})^o$$

$$((A^{cl})^o)^{cl} = ((A^c)^o)^{cl}$$

combining the three equalities we have $(((A^{cl})^o)^{cl})^o = ((A^c)^o)^{cl}$, then take the complement to obtain the result.

(3), (4), (6), (8) follow from applying (1) and (2) repeatedly.

Now we show that $S$ is invariant:

- operations on $A$: we have by definition of $S$, $A^o \in S$, $A^{cl} \in S$ and $A^c \in S$
- operations on $A^o$: $(A^o)^o = A^o \in S$ (the interior of an open set is itself), $(A^o)^{cl} \in S$, $(A^o)^c = (A^c)^{cl} \in S$ by (1)
• operations on $A^\circ$: $(A^\circ)^\circ \in S$, $(A^\circ)^\circ = A^\circ \in S$ (the closure of a closed set is itself), and $(A^\circ)^\circ = (A^\circ)^\circ \in S$ by (2)

• operations on $(A^\circ)^\circ$: $((A^\circ)^\circ)^\circ \in S$, $((A^\circ)^\circ)^\circ = (A^\circ)^\circ \in S$ $(A^\circ)^\circ = ((A^\circ)^\circ)^\circ \in S$ by (3)

• operations on $(A^\circ)^\circ$: $((A^\circ)^\circ)^\circ = (A^\circ)^\circ \in S$, $((A^\circ)^\circ)^\circ = ((A^\circ)^\circ)^\circ \in S$ by (4)

• operations on $((A^\circ)^\circ)^\circ$: $(((A^\circ)^\circ)^\circ)^\circ \in S$, $(((A^\circ)^\circ)^\circ)^\circ = (A^\circ)^\circ \in S$ by (5), $(((A^\circ)^\circ)^\circ)^\circ = (((A^\circ)^\circ)^\circ)^\circ \in S$ by (6)

• operations on $((A^\circ)^\circ)^\circ$: $(((A^\circ)^\circ)^\circ)^\circ = (A^\circ)^\circ \in S$ by (7), $(((A^\circ)^\circ)^\circ)^\circ = ((A^\circ)^\circ)^\circ \in S$, $(((A^\circ)^\circ)^\circ)^\circ = (((A^\circ)^\circ)^\circ)^\circ \in S$ by (8)

• operations on $A$, $(A^\circ)^\circ$, $(A^\circ)^\circ$, $(A^\circ)^\circ$, $(A^\circ)^\circ$, $(A^\circ)^\circ$, $(A^\circ)^\circ$, $(A^\circ)^\circ$, $(A^\circ)^\circ$, $(A^\circ)^\circ$: obtained from the previous statements by applying them to $A^\circ$ instead of $A$, (observing that $S$ is symmetric in $A$ and $A^\circ$).

This shows that one can constructs at most 14 sets starting from $A$ and applying those operations a finite number of times. The following example shows that this upper bound is attained: consider the metric space $(\mathbb{R}, d)$ where $d$ is the usual metric, and let $A = ([0, 1] \cap \mathbb{Q}) \cup [1, 2) \cup (2, 3] \cup \{4\}$. Then we have

• $A^\circ = (1, 2) \cup (2, 3)$

• $A^\circ = [0, 3] \cup \{4\}$

• $(A^\circ)^\circ = [1, 3]$  

• $(A^\circ)^\circ = (0, 3)$

• $((A^\circ)^\circ)^\circ = (1, 3)$

• $(((A^\circ)^\circ)^\circ)^\circ = [0, 3]$  

• $A^\circ = (-\infty, 0) \cup ((\mathbb{R} \setminus \mathbb{Q}) \cap (0, 1)) \cup \{2\} \cup (3, 4) \cup (4, \infty)$

• $(A^\circ)^\circ = (-\infty, 0) \cup (3, 4) \cup (4, \infty)$

• $(A^\circ)^\circ = (-\infty, 0] \cup [3, \infty)$

• $(A^\circ)^\circ = (-\infty, 1] \cup [3, \infty)$

• $(A^\circ)^\circ = (-\infty, 1) \cup (3, \infty)$

• $((A^\circ)^\circ)^\circ = (-\infty, 0) \cup (3, \infty)$

• $(((A^\circ)^\circ)^\circ)^\circ = (-\infty, 1] \cup [3, \infty)$

which are all distinct.