# MATH 202A - Problem Set 2

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**(2.1)** Let $S$ be a nonempty set, $n \in \mathbb{N}$, and $X = S^n$. Let

$$d : (X, X) \to \mathbb{N}$$
$$(x, y) \mapsto d(x, y) = |J_{x,y}|$$

where $J_{x,y} = \{j \in \{1, \dots n\} | x_j \neq y_j\}$. Then $d$ is a metric.

*proof:* We have by definition of $d$, $\forall x \in X$, $d(x, x) \geq 0$, and $\forall x, y \in X$, $d(x, y) = d(y, x)$, which proves reflexivity and symmetry. To show the triangle inequality, consider $x, y, z \in X$. We have $J_{x,y} \subseteq (J_{x,z} \cup J_{y,z})$. Indeed, let $j \in J_{x,y}$, then we have $x_j \neq y_j$, thus we have $x_j \neq z_j$ or $y_j \neq z_j$, i.e. $j \in J_{x,z} \cup J_{y,z}$. Then we have $d(x, y) = |J_{x,y}| \leq |J_{x,z} \cup J_{y,z}| \leq |J_{x,z}| + |J_{y,z}| = d(x, z) + d(y, z)$, which proves the triangle inequality.

**(2.2)**

**(2.3)** Consider the metric space $\mathbb{Q}$ with the metric $d(x, y) = |x - y|$. Define the set $X = \{x \in \mathbb{Q} \mid |x| < \sqrt{2}\}$ is both open and closed.

**(2.4)** Let $(X, d)$ be $\mathbb{R}^2$ with the standard metric $d(a, b) = \|a - b\|_2 = ((a_1 - b_1)^2 + (a_2 - b_2)^2)^{1/2}$. Let $A = \{(x, y) \in \mathbb{Q}^2 : x^2 + y^2 < 1\} = \{a \in \mathbb{Q}^2 : \|a\|_2 < 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} = \{b \in \mathbb{R}^2 : \|b\|_2 \leq 1\}$. Then $\operatorname{bd} A = B$.

*proof:* We use the fact that $\forall b \in B$, $\forall \epsilon > 0$,

(a) $B(\epsilon, b) \cap (A - \{b\}) \neq \emptyset$ and

(b) $B(\epsilon, b) \cap (A^c - \{b\}) \neq \emptyset$

To prove (a), let $b \in B$, and fix $\epsilon > 0$. Let $b' = (1 - \epsilon/2)b$ (this is to handle the case where $b$ lies on the sphere, see figure 1). Then we have $\|b'\|_2 = (1 - \epsilon/2)\|b\|_2 < 1$. Let $\epsilon' = \frac{\epsilon}{2\sqrt{2}} > 0$. Then we have $(b'_1 - \epsilon', b'_1) \cup (b'_1, b'_1 + \epsilon')$ contains elements of $\mathbb{Q}$. Similarly, $(b'_2 - \epsilon', b'_2) \cup (b'_2, b'_2 + \epsilon')$ contains elements of $\mathbb{Q}$. Therefore there exists $c \in \mathbb{Q}^2$ such that $c \neq b'$ such that $|c_1 - b'_1| < \epsilon'$ and $|c_2 - b'_2| < \epsilon'$, and $c$ satisfies $\|c - b'\|_2 = \sqrt{(c_1 - b'_1)^2 + (c_2 - b'_2)^2} < \sqrt{2\epsilon'^2} = \epsilon/2$. And we have using the triangle inequality

$$\begin{aligned}
\|c - b\|_2 &\leq \|c - b'\|_2 + \|b' - b\|_2 \\
&= \|c - b'\|_2 + \|\frac{\epsilon}{2} b\| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}\|b\|_2 \\
&\leq \epsilon
\end{aligned}$$

and

$$
\begin{aligned}
\|c\|_2 &\le \|b'\|_2 + \|c - b'\|_2 \\
&< (1 - \frac{\epsilon}{2})\|b\|_2 + \frac{\epsilon}{2} \\
&= \|b\|_2 + \frac{\epsilon}{2}(1 - \|b\|_2) \\
&\le \|b\|_2 \\
&\le 1
\end{aligned}
$$

Therefore $c \in B(\epsilon, b)$, $c \in A$ and $c \neq b$, i.e. that $c \in B(\epsilon, b) \cap (A - \{b\})$, which proves (a). To prove (b), a similar argument is used, where $c$ is now taken to be in $(\mathbb{R} - \mathbb{Q})^2$.

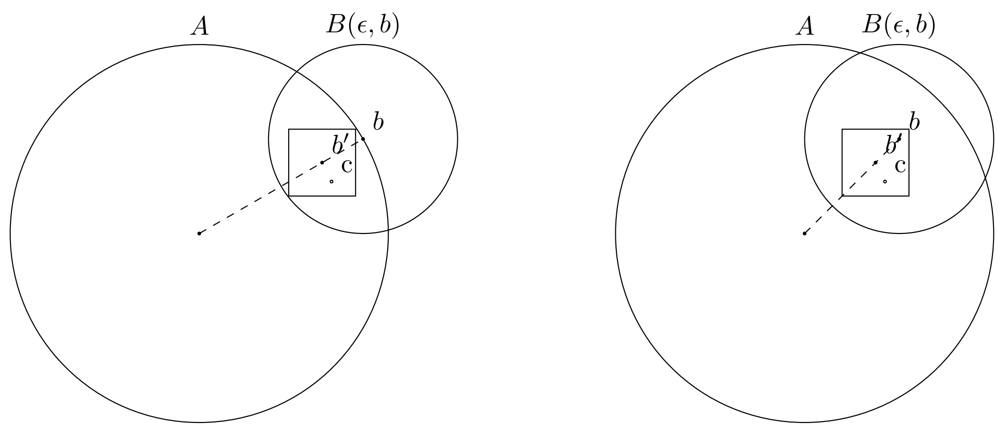



Figure 1: Illustration of the proof of (a) and (b). Left: case where $b$ lies on the unit sphere $\{a \in \mathbb{R}^2 : \|a\|_2 = 1\}$, right: generic case.

*proof of* $B = \operatorname{bd} A$*:* To prove that $B \subseteq \operatorname{bd} A$, let $b \in B$, and construct a sequence $(a_n)$ as follows: $\forall n \in \mathbb{N}$, let $\epsilon = \frac{1}{n+1}$, and define $a_n$ to be any element in $B(\epsilon, b) \cap (A - \{b\})$ (nonempty by (a)). $(a_n)$ thus defined is a sequence of $A - \{b\}$ that converges to $b$, thus $b$ is a limit point of $A$. Now define sequence $(b_n)$ as follows: $\forall n \in \mathbb{N}$, let $\epsilon = \frac{1}{n+1}$, and define $b_n$ to be any element in $B(\epsilon, b) \cap (A^c - \{b\})$ (nonempty by (b)). $(b_n)$ thus defined is a sequence of $A^c - \{b\}$ that converges to $b$, thus $b$ is a limit point of $A^c$. This proves that $B \subseteq \operatorname{bd} A$.

Conversely, if $b \notin B$, then $\|b\|_2 > 1$, and there exists $\epsilon > 0$ such that $B(\epsilon, b) \cap A = \emptyset$ (for example take $\epsilon = 1 - \|b\|_2$). Thus $b$ is not a limit point of $A$, and as a consequence $b \notin \operatorname{bd} A$.

**(2.5)** Let $A, B$ be subsets of a metric space. Then $A^o \cap B^o = (A \cap B)^o$.

*proof:* $\subseteq$ Let $x \in A^o \cap B^o$. Since $x \in A^o$, $\exists \epsilon_1 > 0$ such that $B(\epsilon_1, x) \subseteq A$, and since $x \in B^o$, $\exists \epsilon_2 > 0$ such that $B(\epsilon_2, x) \subseteq B$. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$. Then we have $B(\epsilon, x) \subseteq A \cap B$, thus $x \in (A \cap B)^o$.

$\supseteq$ Let $x \in (A \cap B)^o$. Then exists $\epsilon > 0$ such that $B(\epsilon, x) \subseteq (A \cap B)$. Thus we have $B(\epsilon, x) \subseteq A$ and $B(\epsilon, x) \subseteq B$, thus $x \in A^o$ and $x \in B^o$, therefore $x \in A^o \cap B^o$.

**(2.6)** Let $A = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, \text{ and } xy > 1\}$. Then $A$ is an open set.

*proof:* Consider the metric

$$
\begin{aligned}
d &: \mathbb{R}^2 \to \mathbb{R} \\
(a, b) &\mapsto \|a - b\|_1 = |a_1 - b_1| + |a_2 - b_2|
\end{aligned}
$$

Let $f : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto x$, $g : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto x_2$, and $h : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto x_1 x_2$. The function $f$ is continuous since $\forall a, b \in \mathbb{R}^2$, we have

$$
d(f(a), f(b)) = d(a_1, b_1) = |a_1 - b_1| \le |a_1 - b_1| + |a_2 - b_2| = d(a, b)
$$

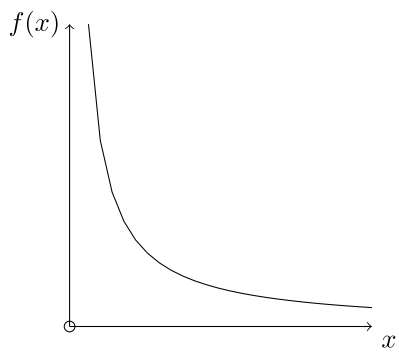
therefore $\forall \epsilon > 0$, $d(a,b) \leq \epsilon \Rightarrow d(f(a), f(b)) \leq \epsilon$ and $f$ is continuous. Similarly, $g$ is continuous. Finally, $h$ is continuous as the product of continuous functions $f$ and $g$[1].

Then we have $A = f^{-1}((0,+\infty)) \cap g^{-1}((0,+\infty)) \cap h^{-1}((1,+\infty))$ is the intersection of open sets (inverse image by continuous functions of open sets) and is therefore open.

**(2.7)** Consider $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{if } x > 0 \end{cases}$$

The graph of $f$, $G = \{(x, 1/x), x \in (0,1]\} \cup (0,0)$ is closed, but $f$ is not continuous.




**(2.8)** Let $(x_n)$ be a sequence of points in a metric space $(X,d)$, and let $z \in X$. Suppose that any subsequence of $(x_n)$ has a subsequence that converges to $z$. Then $\lim_{n\to\infty} x_n = z$. *proof:* by contrapositive. Assume $(x_n)$ does not converge to $z$. Then $\exists \epsilon > 0$ such that

$$\forall N \in \mathbb{N},\ \exists n \geq N \text{ such that } d(x_n, z) > \epsilon \tag{1}$$

We use this fact to construct a subsequence $(x_{\phi(n)})_n$ ($\phi : \mathbb{N} \to \mathbb{N}$ strictly increasing) such that $\forall n \in \mathbb{N}$, $d(x_{\phi(n)} - z) > \epsilon$. The subsequence $(x_{\phi(n)})$ is defined by induction:

- We know by (1) (taking $N = 0$) that $\exists n_0 \geq 0$ such that $d(x_{n_0}, z) > \epsilon$. Define $\phi(0) = n_0$
- Assume $(\phi(n))$ defined. Then we know by (1) (taking $N = \phi(n)+1$) that $\exists n_1 \geq \phi(n)+1$ such that $d(x_{n_1}, z) > \epsilon$. Define $\phi(n+1) = n_1$. We have $\phi(n+1) > \phi(n)$.

By construction, the subsequence $(x_{\phi(n)})_n$ satisfies $\forall n \in \mathbb{N}$, $d(x_{\phi(n)}, z) > \epsilon$, therefore does not have a subsequence that converges to $z$. This proves the contrapositive of the desired result.

[1] proof that the product of continuous functions is continuous: let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ continuous at $x_0$, where $(X,d)$ is a metric space. Let $\epsilon > 0$. Since $f$ and $g$ are continuous, $\forall \epsilon' > 0$, $\exists \delta_1 > 0, \delta_2 > 0$ such that $d(x,x_0) \leq \delta \Rightarrow |f(x) - f(x_0)| \leq \epsilon'$, and $d(x,x_0) \leq \delta \Rightarrow |g(x) - g(x_0)| \leq \epsilon'$

$$\begin{aligned} |f(x)g(x) - f(x_0)g(x_0)| &= |(f(x) - f(x_0) + f(x_0))(g(x) - g(x_0) + g(x_0)) - f(x_0)g(x_0)| \\ &= |(f(x) - f(x_0))(g(x) - g(x_0)) + (f(x) - f(x_0))g(x_0) + f(x_0)(g(x) - g(x_0))| \\ &\leq |f(x) - f(x_0)||g(x) - g(x_0)| + |f(x) - f(x_0)||g(x_0)| + |f(x_0)||g(x) - g(x_0)| \end{aligned}$$

thus if $d(x,x_0) \leq \min(\delta_1, \delta_2)$,

$$|f(x)g(x) - f(x_0)g(x_0)| \leq \epsilon'^2 + \epsilon'(|g(x_0)| + |f(x_0)|)$$

taking $\epsilon'$ small enough, we can ensure $|f(x)g(x) - f(x_0)g(x_0)| \leq \epsilon$

**(2.9)** Let $A$ be a subset of a metric space $(X,d)$, and let

$$S = \{A, A^o, A^{cl}, (A^o)^{cl}, (A^{cl})^o, ((A^o)^{cl})^o, ((A^{cl})^o)^{cl},$$
$$A^c, (A^c)^o, (A^c)^{cl}, ((A^c)^o)^{cl}, ((A^c)^{cl})^o, (((A^c)o)^{cl})^o, (((A^c)^{cl})^o)^{cl}\}$$

We show that $S$ is invariant by the following operations: interior, closure and complement. We first show some facts:

$$(A^o)^c = (A^c)^{cl} \tag{1}$$
$$(A^{cl})^c = (A^c)^o \tag{2}$$
$$((A^o)^{cl})^c = ((A^c)^{cl})^o \tag{3}$$
$$((A^{cl})^o)^c = ((A^c)^o)^{cl} \tag{4}$$
$$(((A^o)^{cl})^o)^{cl} = (A^o)^{cl} \tag{5}$$
$$(((A^o)^{cl})^o)^c = (((A^c)^{cl})^o)^{cl} \tag{6}$$
$$(((A^{cl})^o)^{cl})^o = (A^{cl})^o \tag{7}$$
$$(((A^{cl})^o)^{cl})^c = (((A^c)^o)^{cl})^o \tag{8}$$

*proof:*
(1) we have

$$\begin{aligned} x \in (A^o)^c &\Leftrightarrow \forall \epsilon > 0, B(\epsilon, x) \nsubseteq A \\ &\Leftrightarrow \forall \epsilon > 0, B(\epsilon, x) \cap A^c \neq \emptyset \\ &\Leftrightarrow x \in (A^c)^{cl} \end{aligned}$$

(5) We have $A^o \subseteq (A^o)^{cl}$ by definition of the closure. Thus by monotonicity of the interior, $(A^o)^o \subseteq ((A^o)^{cl})^o$, i.e. $A^o \subseteq ((A^o)^{cl})^o$, and by monotonicity of the closure, $(A^o)^{cl} \subseteq (((A^o)^{cl})^o)^{cl}$.

To prove the reverse inclusion, we have by definition of the interior, $((A^o)^{cl})^o \subseteq (A^o)^{cl}$, thus taking the closure, $(((A^o)^{cl})^o)^{cl} \subseteq ((A^o)^{cl})^{cl} = (A^o)^{cl}$

(2) replacing $A$ with $A^c$ in (1), we have $((A^c)^o)^c = A^{cl}$, thus taking the complement, we obtain (2)

(7) replacing $A$ with $A^c$ in (5), we have

$$((((A^c)^o)^{cl})^o)^{cl} = ((A^c)^o)^{cl}$$

and applying identities (1) and (2) repeatedly, we have

$$\begin{aligned} ((((A^c)^o)^{cl})^o)^{cl} &= ((((A^{cl})^o)^{cl})^o)^c \\ ((A^c)^o)^{cl} &= ((A^{cl})^o)^c \end{aligned}$$

combining the three equalities we have $((((A^{cl})^o)^{cl})^o)^c = ((A^{cl})^o)^c$, then take the complement to obtain the result.

(3), (4), (6), (8) follow from applying (1) and (2) repeatedly.

Now we show that $S$ is invariant:

- operations on $A$: we have by definition of $S$, $A^o \in S$, $A^{cl} \in S$ and $A^c \in S$
- operations on $A^o$: $(A^o)^o = A^o \in S$ (the interior of an open set is itself), $(A^o)^{cl} \in S$, $(A^o)^c = (A^c)^{cl} \in S$ by (1)

- operations on $A^{cl}$: $(A^{cl})^o \in S$, $(A^{cl})^{cl} = A^{cl} \in S$ (the closure of a closed set is itself), and $(A^{cl})^c = (A^c)^o \in S$ by (2)
- operations on $(A^o)^{cl}$: $((A^o)^{cl})^o \in S$, $((A^o)^{cl})^{cl} = (A^o)^{cl} \in S$, $((A^o)^{cl})^c = ((A^c)^{cl})^o \in S$ by (3)
- operations on $(A^{cl})^o$: $((A^{cl})^o)^o = (A^{cl})^o \in S$, $((A^{cl})^o)^{cl} \in S$, $((A^{cl})^o)^c = ((A^c)^o)^{cl} \in S$ by (4)
- operations on $((A^o)^{cl})^o$: $(((A^o)^{cl})^o)^o = ((A^o)^{cl})^o \in S$, $(((A^o)^{cl})^o)^{cl} = (A^o)^{cl} \in S$ by (5), $(((A^o)^{cl})^o)^c = (((A^c)^{cl})^o)^{cl} \in S$ by (6)
- operations on $((A^{cl})^o)^{cl}$: $(((A^{cl})^o)^{cl})^o = (A^{cl})^o \in S$ by (7), $(((A^{cl})^o)^{cl})^{cl} = ((A^{cl})^o)^{cl} \in S$, $(((A^{cl})^o)^{cl})^c = (((A^c)^o)^{cl})^o \in S$ by (8)
- operations on $A^c, (A^c)^o, (A^c)^{cl}, ((A^c)^o)^{cl}, ((A^c)^{cl})^o, (((A^c)o)^{cl})^o, (((A^c)^{cl})^o)^{cl}$: obtained from the previous statements by applying them to $A^c$ instead of $A$, (observing that $S$ is symmetric in $A$ and $A^c$).

This shows that one can constructs at most 14 sets starting from $A$ and applying those operations a finite number of times. The following example shows that this upper bound is attained: consider the metric space $(\mathbb{R}, d)$ where $d$ is the usual metric, and let $A = ([0,1] \cap \mathbb{Q}) \cup [1,2) \cup (2,3] \cup \{4\}$. Then we have

- $A^o = (1,2) \cup (2,3)$
- $A^{cl} = [0,3] \cup \{4\}$
- $(A^o)^{cl} = [1,3]$
- $(A^{cl})^o = (0,3)$
- $((A^o)^{cl})^o = (1,3)$
- $((A^{cl})^o)^{cl} = [0,3]$
- $A^c = (-\infty, 0) \cup ((\mathbb{R} - \mathbb{Q}) \cap (0,1)) \cup \{2\} \cup (3,4) \cup (4,\infty)$
- $(A^c)^o = (-\infty,0) \cup (3,4) \cup (4,\infty)$
- $((A^c)^o)^{cl} = (-\infty,0] \cup [3,\infty)$
- $(A^c)^{cl} = (-\infty,1] \cup \{2\} \cup [3,\infty)$
- $((A^c)^{cl})^o = (-\infty,1) \cup (3,\infty)$
- $(((A^c)^o)^{cl})^o = (-\infty,0) \cup (3,\infty)$
- $(((A^c)^{cl})^o)^{cl} = (-\infty,1] \cup [3,\infty)$

which are all distinct.