(13.1) Let \( y \in \mathbb{R} \). Then
1. \( E \) is a Borel set if and only if \( y + E \) is a Borel set
2. \( E \) is Lebesgue measurable if and only if \( y + E \) is Lebesgue measurable.

**proof**
1. Let \( z \in \mathbb{R} \), and consider the function
   \[
   f_z : \mathbb{R} \to \mathbb{R} \\
   x \mapsto x + z
   \]
   We have \( f_z \) is Borel-measurable since for all \( c \in \mathbb{R} \), \( \{ x \in \mathbb{R} | f_z(x) \leq c \} = \{ x \in \mathbb{R} | x + z \leq c \} = (\infty, c - z) \in \mathscr{B} \). We also observe that if \( E \subseteq \mathbb{R} \), then \( f_{-y}^{-1}(E) = \{ x \in \mathbb{R} | x + z \in E \} = \{ -z + (x + z) | x + z \in E \} = -z + E \). Thus we have: if \( E \) is a Borel set, then \( f_{-y}^{-1}(E) = y + E \) is a Borel set. Conversely, if \( y + E \) is a Borel set, then \( -y + (y + E) = E \) is a Borel set. This proves the equivalence.
2. Let \( E \) be a Lebesgue measurable set. Then \( E = E' \Delta Z \) for some \( E' \in \mathcal{B} \) and \( Z \subseteq Z' \in \mathcal{B} \) with \( m(Z') = 0 \). Then we have
   \[
   y + E = f_{-y}^{-1}E
   = f_{-y}^{-1}(E' \Delta Z)
   = f_{-y}^{-1}(E') \Delta f_{-y}^{-1}(Z)
   = (y + E') \Delta (y + Z)
   \]
   where \( y + E' \in \mathcal{B} \), and \( y + Z \subseteq y + Z' \in \mathcal{B} \), and \( m(y + Z') = m(Z') = 0 \). Therefore \( y + E \) is Lebesgue measurable.
   Conversely, if \( y + E \) is Lebesgue measurable, then \( -y + (y + E) = E \) is Lebesgue measurable. This proves the equivalence.

(13.2) Let \( E \subseteq \mathbb{R} \) be a Borel set. Then the function
   \[
   F : \mathbb{R} \to \mathbb{R}^* \\
   t \mapsto F(t) = m(E \cap (-\infty, t])
   \]
is continuous on \( \mathbb{R} \).

**proof** Let \( t \in \mathbb{R} \), and let \((t_n)\) be any sequence of reals that converges to \( t \). Then we have for all \( n \in \mathbb{N} \), if \( t_n \leq t \), then \((\infty, t] = (\infty, t_n] \cup (t_n, t)\) disjointly, then taking the measure, we have \( F(t) = F(t_n) + |t - t_n| \).
Similarly, if \( t \leq t_n \), then \((\infty, t_n] = (\infty, t] \cup (t, t_n)\) disjointly, thus \( F(t_n) = F(t) + |t_n - t| \). Therefore we have in both cases
   \[
   F(t_n) = F(t) + \epsilon_n
   \]
where \( |\epsilon_n| = |t_n - t| \). We have \( \epsilon_n \to 0 \), thus \( (F(t_n))_n \) converges (in \( \mathbb{R}^* \)), and \( \lim_{n \to \infty} F(t_n) = F(t) \). This proves that \( F \) is continuous at \( t \). Since this holds for any \( t \), \( F \) is continuous on \( \mathbb{R} \).
(13.3) Let $E$ be a Lebesgue measurable set such that $m(E) > 0$. Then $E$ contains a subset which is not Lebesgue measurable.

**proof** Suppose, by contradiction, that all subset of $E$ are Lebesgue measurable.

Consider the equivalence relation on $\mathbb{R} \times \mathbb{R}$: $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Then let $\mathcal{E}$ be a set that contains one and only one element of each equivalence class$^1$. $\mathcal{E}$ satisfies the following properties:

1. for all $y \in \mathbb{R}$, there exists $x \in \mathcal{E}$ such that $x - y \in \mathbb{Q}$; i.e. there exists $x \in \mathcal{E}$ and $q \in \mathbb{Q}$ such that $y = q + x$. Therefore
   \[ \mathbb{R} = \cup_{q \in \mathbb{Q}} (q + \mathcal{E}) \]

2. $x, x' \in \mathcal{E}$ implies $x - x' \notin \mathbb{Q}$. Thus if $p, q$ are distinct rationals, then $p + \mathcal{E}$ and $q + \mathcal{E}$ are disjoint. Indeed, if $x \in p + \mathcal{E}$ and $x' \in q + \mathcal{E}$, then $x - p$ and $x' - q$ are elements of $\mathcal{E}$, thus $x - p - (x') - q \notin \mathbb{Q}$, thus $x - x' \notin \mathbb{Q}$, in particular, $x - x' \neq 0$.

Now we have $E = \mathbb{R} \cap \mathcal{E} = (\cup_{q \in \mathbb{Q}} (q + \mathcal{E})) \cap \mathcal{E} = \cup_{q \in \mathbb{Q}} ((q + \mathcal{E}) \cap \mathcal{E})$, and the union is disjoint. Therefore by $\sigma$-additivity of the Lebesgue measure $m$, we have

\[ m(E) = \sum_{q \in \mathbb{Q}} m((q + \mathcal{E}) \cap \mathcal{E}) \]

and since $m(E) > 0$, there exists $q_0 \in \mathbb{Q}$ such that $m(E_0 \cap \mathcal{E}) > 0$ where $E_0 = q_0 + \mathcal{E}$. Next, we have $\mathbb{R} = \cup_{n \in \mathbb{Z}} [n, n + 1]$ disjointly, thus taking the intersection with $E_0 \cap \mathcal{E}$ and using $\sigma$-additivity of $m$, we have

\[ m(E) = \sum_{n \in \mathbb{Z}} m([n, n + 1] \cap E_0 \cap \mathcal{E}) \]

and since $m(E) > 0$, there exists $n_0 \in \mathbb{Z}$ such that $m([n_0, n_0 + 1] \cap E_0 \cap \mathcal{E}) > 0$. Finally, we have

\[ \cup_{q \in \mathbb{Q} \cap [0, 1]} q + ([n_0, n_0 + 1] \cap (q_0 + \mathcal{E}) \cap \mathcal{E}) \subseteq [n_0, n_0 + 2] \]

but this union is disjoint by the second property of $\mathcal{E}$, therefore taking the measure, we have

\[ \sum_{q \in \mathbb{Q} \cap [0, 1]} m(q + [n_0, n_0 + 1] \cap (q_0 + \mathcal{E}) \cap \mathcal{E}) \leq 2 \]

and using the fact that $m(x + A) = m(A)$ for any measurable subset $A$, we have that each term in the sum is equal to $m([n_0, n_0 + 1] \cap (q_0 + \mathcal{E}) \cap \mathcal{E})$, and since the sum has an infinite number of terms and $m([n_0, n_0 + 1] \cap (q_0 + \mathcal{E}) \cap \mathcal{E}) > 0$, we have

\[ +\infty \leq 2 \]

which is a contradiction. This proves the result.

---

$^1$Such a set can be obtained as the image of the quotient set $\mathbb{R}/\sim$ by an injection $i : \mathbb{R}/\sim \to \mathbb{R}$, and such an injection exists by the axiom of choice and the fact that the function

\[ s : \mathbb{R} \to \mathbb{R}/\sim \\
\quad x \mapsto [x] \]

is a surjection.
Let $(X, \mathcal{A}, \mu)$ be a measure space, and $(X, \bar{\mathcal{A}}, \bar{\mu})$ its completion. Let $f : X \to \mathbb{R}^*$ be a measurable function with respect to $\bar{\mathcal{A}}$. Then there exists a function $g : X \to \mathbb{R}^*$ measurable with respect to $\mathcal{A}$, with $g = f$ $\bar{\mu}$-almost everywhere.

**proof** For all $q \in \mathbb{Q}$, let

\[ E_q = f^{-1}((\infty, q]) \]

since $f$ is measurable with respect to $\bar{\mathcal{A}}$, for all $q \in \mathbb{Q}$, $E_q \in \bar{\mathcal{A}}$, thus there exists $A_q \in \mathcal{A}$ such that $A_q \subseteq E_q$, and $\bar{\mu}(E_q \setminus A_q) = 0$.\(^2\) We observe that $\forall q, q' \in \mathbb{Q}$, $q \leq q'$ $\Rightarrow$ $E_q \subseteq E_{q'}$, and

\[ \forall x \in X, \ f(x) = \inf\{q \in \mathbb{Q}| f(x) \leq q \} = \inf\{q \in \mathbb{Q}| x \in E_q \} \]

Now for all $q \in \mathbb{Q}$, define the set

\[ \bar{A}_q = \bigcup_{r \in \mathbb{Q}, r \leq q} A_r \]

Then $\bar{A}_q \in \mathcal{A}$ since it is the union of countably many measurable sets. We also have $\forall q, q' \in \mathbb{Q}$, $q \leq q' \Rightarrow \bar{A}_q \subseteq \bar{A}_{q'}$.

Now we define the function $g$:

\[ g : X \to \mathbb{R}^* \]
\[ x \mapsto g(x) = \inf\{q \in \mathbb{Q}| x \in \bar{A}_q \} \]

with the convention $\inf \emptyset = +\infty$. We observe that $g$ can also be written as a pointwise infimum of measurable functions:

\[ g(x) = \inf_{q \in \mathbb{Q}} g_q(x) \]

where for all $q \in \mathbb{Q}$, $g_q$ is defined by

\[ g_q : X \to \mathbb{R}^* \]
\[ x \mapsto g_q(x) = \begin{cases} q & \text{if } x \in \bar{A}_q \\ +\infty & \text{otherwise} \end{cases} \]

$g_q$ is measurable as the sum of two measurable functions, $g_q = q1_{\bar{A}_q} + +\infty1_{\bar{A}_q^c}$, where $\bar{A}_q$ and its complements are elements of $\mathcal{A}$. Therefore $g$ is $\mathcal{A}$-measurable as the pointwise infimum of countably many $\mathcal{A}$-measurable functions.

Finally, we have for all $q \in \mathbb{Q}$,

\[ g(x) \leq q \iff q \in \bar{A}_q \]

indeed, we have

- if $g(x) \leq q$, then by definition of the inf, for all $n \geq 1$, there exists $p_n \in \{g(x), g(x) + 1/n\}$ such that $x \in \bar{A}_{p_n}$. But $p_n \leq g(x) + 1/n \leq q + 1/n$, thus $\bar{A}_{p_n} \subseteq \bar{A}_{q+1/n}$. Therefore for all $n \geq 1$, $x \in \bar{A}_{q+1/n}$, thus

\[ x \in \cap_{n \geq 1} \bar{A}_{q+1/n} = \bar{A}_q \]

\(^2\)Indeed, there exist measurable sets $E'_q, S'_q, T'_q$, subsets $S_q \subseteq S'_q$ and $T_q \subseteq T'_q$, such that $E_q = (E'_q \cup S_q) \setminus T_q$. Then if we let

\[ A_q = E'_q \setminus T'_q \]

then we have $A_q$ is measurable, $A_q \subseteq E_q$, and $E_q \setminus A_q \subseteq S'_q \cup T'_q$, thus

\[ \bar{\mu}(E_q \setminus A_q) \leq \bar{\mu}(S'_q) + \bar{\mu}(T'_q) = \mu(S'_q) + \mu(T'_q) = 0 \]
• if \( x \in \tilde{A}_q \), then \( q \in \{ p \in \mathbb{Q} | x \in \tilde{A}_p \} \) therefore taking the inf, \( g(x) \leq q \). This proves the claim.

We have for all \( q \in \mathbb{Q} \), and for all rational \( r \leq q \), \( A_r \subseteq A_q \subseteq E_q \), therefore

\[
\tilde{A}_q = \bigcup_{r \in \mathbb{Q}, r \leq q} A_r \subseteq E_q
\]

therefore we have for all \( x \in X \), \( \inf \{ q \in \mathbb{Q} | x \in E_q \} \leq \inf \{ q \in \mathbb{Q} | x \in \tilde{A}_q \} \), i.e.

\[
f(x) \leq g(x)
\]

and we have

\[
f(x) \neq g(x) \Rightarrow f(x) < g(x)
\]

\[
\Rightarrow \exists q \in \mathbb{Q} : f(x) \leq q < g(x)
\]

\[
\Rightarrow \exists q \in \mathbb{Q} : x \in E_q \cap \tilde{A}_q^c
\]

\[
\Rightarrow x \in \bigcup_{q \in \mathbb{Q}} E_q \setminus \tilde{A}_q
\]

Therefore if \( D = \{ x \in X | f(x) \neq g(x) \} \), we have

\[
\bar{\mu}(D) \leq \bar{\mu}(\bigcup_{q \in \mathbb{Q}} E_q \setminus \tilde{A}_q)
\]

\[
\leq \sum_{q \in \mathbb{Q}} \bar{\mu}(E_q \setminus \tilde{A}_q)
\]

but \( E_q \setminus \tilde{A}_q \subseteq E_q \setminus A_q \) (since \( A_q \subseteq \tilde{A}_q \)) therefore for all \( q \), \( \bar{\mu}(E_q \setminus \tilde{A}_q) = 0 \). Therefore \( \bar{\mu}(D) = 0 \), and \( f \) and \( g \) disagree on a set of measure 0.

(13.5) Let \( X \) be a set, and \( \mathcal{A} = P(X) \) be the \( \sigma \)-algebra of its subsets. Let \( f_1, \ldots, f_N : X \rightarrow \mathbb{R} \) be a finite collection of measurable functions, and \( h : \mathbb{R}^N \rightarrow \mathbb{C} \) be a continuous function. Then the function

\[
g : X \rightarrow \mathbb{C}
\]

\[
x \mapsto h(f_1(x), \ldots, f_N(x))
\]

is measurable (i.e. for all open sets \( O \subseteq \mathbb{C} \), \( g^{-1}(O) \) is a measurable subset of \( \mathbb{R} \))

**proof** Let \( F \) be the function

\[
F : X \rightarrow \mathbb{R}^n
\]

\[
x \mapsto (f_1(x), \ldots, f_n(x))
\]

We first prove that \( F \) is measurable. We first observe that any open set in \( \mathbb{R}^n \) is the union of countably many basic open sets of the form \( U_1 \times U_n \), where for all \( i \), \( U_i \) is open. Therefore it suffices to show that the inverse image by \( F \) of basic open sets. Let \( U = U_1 \times \cdots \times U_n \) be a basic open set. Then we have

\[
x \in F^{-1}(U) \iff (f_1(x), \ldots, f_N(x)) \in U
\]

\[
\iff f_i(x) \in U_i \forall i \in \{1, \ldots, N\}
\]

\[
\iff x \in \bigcap_{i=1}^N f_i^{-1}(U_i)
\]

where for all \( i \), \( f_i^{-1}(U_i) \) is measurable since \( f_i \) is measurable and \( U_i \) is open. Therefore \( F \) is measurable.

Now let \( O \) be an open subset of \( \mathbb{C} \). We have \( g = h \circ F \), therefore \( g^{-1}(O) = F^{-1}(h^{-1}(O)) \). Since \( h \) is continuous and \( O \) is open, we have \( h^{-1}(O) \) is open. Then since \( F \) was proven to be measurable, \( F^{-1}(h^{-1}(O)) \) is measurable. This proves that \( g \) is measurable.
Then we have:

$$\forall s \in S, \exists F \supseteq E$$

Let \((X, \mathcal{A}, \mu)\) be a nonatomic measure space, and let \(E \in \mathcal{A}\) such that \(\mu(E) > 0\). Then for all \(t \in [0, \mu(E)]\), there exists a measurable set \(F \subseteq E\) such that \(\mu(F) = t\).

**Proof.** We seek to find a function \(s : [0, \mu(E)] \to \mathcal{A} \cap P(E)\) such that \(s(t)\) is a measurable subset that satisfies \(\mu(s(t)) = t\). To show existence of such a function, consider the set \(\mathcal{S}\) of monotone non-decreasing functions \(s : D_s \to \mathcal{A}\) (where the ordering on \(\mathcal{A}\) is set inclusion) defined on a subset \(D_s \subseteq [0, \mu(E)]\), and such that for all \(t \in D_s\), \(\mu(s(t)) = t\).

For every \(s \in \mathcal{S}\), let \(\mathcal{G}_s = \{(t, s(t)), t \in D_s\}\) be the graph of \(s\). Then equip \(\mathcal{S}\) with the partial ordering \(\leq\) defined by:

\[
\forall s, s' \in \mathcal{S}, s \leq s' \iff \mathcal{G}_s \subseteq \mathcal{G}_{s'}
\]

Then \(\mathcal{S}\) contains the function \(s_0 : \{0\} \to \mathcal{A}\)

\[
s_0(s_0) = \{0\}
\]

therefore \(\mathcal{S}\) is nonempty.

For every chain \(C = \{s_\alpha\}_{\alpha \in A}\) of \(\mathcal{S}\) (a totally ordered subset of \(\mathcal{S}\)), the function \(s\) given by its graph

\[
\mathcal{G}_s = \cup_{\alpha \in A} \mathcal{G}_{s_\alpha}
\]

is an element of \(\mathcal{S}\) and an upper bound of \(C\):

- the function \(s\) is well defined since by definition, the graphs \(\{\mathcal{G}_{s_\alpha}\}\) are totally ordered (with inclusion), therefore their union is a graph.
- the domain of \(s\) is \(D_s = \cup_{\alpha \in A} D_{s_\alpha}\) and is a subset of \([0, \mu(E)]\) since each \(D_{s_\alpha}\) is a subset of \([0, \mu(E)]\).
- for all \(t \in D_s\), there exists \(\alpha \in A\) such that \(t \in D_{s_\alpha}\) (since \(D_s = \cup_{\alpha \in A} D_{s_\alpha}\)), and we have

\[
\mu(s(t)) = \mu(s_\alpha(t)) = t
\]

- for all \(t, t' \in D_s\) such that \(t \leq t'\), there exist \(\alpha, \alpha' \in A\) such that \(t \in D_{s_\alpha}\) and \(t' \in D_{s_{\alpha'}}\). Since the chain is totally ordered, we have either \(s_\alpha \leq s_{\alpha'}\) or \(s_\alpha \leq s_{\alpha'}\). Assume without loss of generality that \(s_{\alpha'} \leq s_\alpha\). Then \(D_{s_{\alpha'}} \subseteq D_{s_\alpha}\) and we have \(t, t'\) are both in \(D_{s_\alpha}\). Thus \(s_\alpha(t) \subseteq s_\alpha(t')\) (since \(s_\alpha\) is monotone non-decreasing), but \(s(t) = s_\alpha(t)\) and \(s(t') = s_\alpha(t')\), therefore

\[
s(t) \leq s(t')
\]

and \(s\) is monotone non-decreasing.

Therefore \(s\) is an element of \(\mathcal{S}\). It is, by definition, an upper bound on the chain \(C\).

Therefore by Zorn’s Lemma, \(\mathcal{S}\) has a maximal element \(s^{\text{max}}\). Next, we show that the domain of \(s^{\text{max}}\), \(D^{s^{\text{max}}}\) is \([0, \mu(E)]\). Suppose by contradiction that this is not the case. Let \(D^{s^{\text{max}}} = [0, \mu(E)] \setminus D^{s^{\text{max}}}\), and consider two cases:

1. if \(D^{s^{\text{max}}}\) is relatively open in \([0, \mu(E)]\), then it is a union of disjoint open intervals (in \([0, \mu(E)]\)). Let \(t_1 < t_2\) be the boundaries of one such open interval. Then we have \(t_1, t_2 \in D^{s^{\text{max}}}\). Let \(F_1 = s^{\text{max}}(t_1)\) and \(F_2 = s^{\text{max}}(t_2)\). Then we have \(F_1 \subseteq F_2\), and they are both measurable subsets of \(E\). Now we have \(F_2 \setminus F_1\) is measurable with measure

\[
\mu(F_2 \setminus F_1) = \mu(F_2) - \mu(F_1) = t_2 - t_1 > 0
\]
therefore, since the space is nonatomic, there exists a measurable \( F \) subset of \( F_2 \setminus F_1 \) with measure \( \mu(F) \in (0, t_2 - t_1) \). Then we have \( F_1 \cup F \) is measurable, is a subset of \( E \), and has measure
\[
\mu(F_1 \cup F) = \mu(F_1) + \mu(F) = t_1 + \mu(F) \\
\in (t_1, t_2)
\]

let \( t_0 = t_1 + \mu(F) \). Then we can extend the function \( s^{\text{max}} \) as follows
\[
\bar{s} : D_{s^{\text{max}}} \cup \{t_0\} \rightarrow A
\]
where \( \bar{s} \) coincides with \( s \) on \( D_{s^{\text{max}}} \), and \( \bar{s}(t_0) = F_1 \cup F \). Then \( \bar{s} \) is still an element of \( \mathcal{S} \), and \( s^{\text{max}} < \bar{s} \), which contradicts maximality of \( s^{\text{max}} \).

2. if \( D_{s^{\text{max}}} \) is not relatively open in \([0, \mu(E)]\), then there exists \( t_0 \in D_{s^{\text{max}}} \) such that for all \( \epsilon > 0 \), \((t_0 - \epsilon, t_0 + \epsilon)\) has nonempty intersection with \( D_{s^{\text{max}}} \). Then construct a sequence \((t_n)\) of elements of \( D_{s^{\text{max}}} \) such that \((|t_n - t_0|)_{n \geq 1}\) is non-increasing and converges to 0. Then we can extract a subsequence of elements that are on one side of \( t_0 \) (this is possible since at least one side has to contain an infinite number of elements). Therefore, assume that \((t_n)_{n \geq 1}\) is a sequence of elements of \( D_{s^{\text{max}}} \), such that \( t_n \leq t_0 \) for all \( n \) (the case \( t_n \geq t_0 \) is treated similarly). In particular \((t_n)\) is non-decreasing and converges to \( t_0 \).

Now let \( F_n = s^{\text{max}}(t_n) \). Then we have for all \( n \), \( t_n \leq t_{n+1} \), thus \( F_n \subseteq F_{n+1} \subseteq E \). Therefore we have \( F = \bigcup_{n \geq 1} F_n \) is a measurable subset of \( E \), and by \( \sigma \)-additivity of \( m \),
\[
\mu(\bigcup_{n} F_n) = \lim_{n \to \infty} \mu(F_n) = \lim_{n \to \infty} t_n = t_0
\]

therefore we can extend \( s^{\text{max}} \) by defining \( \bar{s}(t_0) = \mu(\bigcup_{n \geq 1} F_n) \). Then \( \bar{s} \) is still an element of \( \mathcal{S} \) and \( s^{\text{max}} < \bar{s} \), which contradicts maximality of \( s^{\text{max}} \).

Both cases lead to a contradiction, therefore \( D_{s^{\text{max}}} = [0, \mu(E)] \).

Finally, this provides the desired function
\[
s : [0, \mu(E)] \rightarrow A \cap P(E)
\]
which satisfies for all \( t \in [0, \mu(E)] \), \( F = s(t) \) is a measurable subset of \( E \) such that \( \mu(F) = \mu(s(t)) = t \).
Product measures. Let $(\mathbb{R}, \mathcal{B}, m)$ be the real line with Lebesgue measure on the Borel sets. Let $(X, \mathcal{A}, \mu)$ be any measure space. Let $f : X \to [0, \infty)$ be measurable with respect to $\mathcal{A}$, and consider the graph $G$ of $f$, and the region $\mathcal{R}$ under the graph of $f$:

\[
\mathcal{R} = \{(x, t) \in X \times \mathbb{R} | 0 \leq t < f(x)\} \\
\mathcal{G} = \{(x, t) \in X \times \mathbb{R} | t = f(x)\}
\]

Then

1. $\mathcal{R}$ is a measurable subset of $(X \times \mathbb{R}, \mathcal{A} \times \mathcal{B})$
2. The measure of $\mathcal{R}$ is $\mu \times m(\mathcal{R}) = \int_X f d\mu$
3. $\mathcal{G}$ is measurable (in $\mathcal{A} \times \mathcal{B}$), and if $\mu$ is $\sigma$-finite then $\mu(\mathcal{G}) = 0$ (thus with suitable interpretations, the integral of $f$ is the area under its graph)

**proof**

1. We have

\[
(x, t) \in \mathcal{R} \iff 0 \leq t < f(x) \\
\iff \exists q \in \mathbb{Q} : 0 \leq t \leq q < f(x) \\
\iff \exists q \in \mathbb{Q} : (x, t) \in f^{-1}((q, +\infty)) \times [0, q]
\]

Therefore

\[
\mathcal{R} = \bigcup_{q \in \mathbb{Q}} f^{-1}((q, +\infty)) \times [0, q]
\]

and for all $q \in \mathbb{Q}$, $f^{-1}((q, +\infty)) \times [0, q]$ is measurable in the product space. Therefore $\mathcal{R}$ is the union of countably many measurable sets and is measurable.

2. First, consider the case where $f = s$ is a non-negative simple function. Then there exists a partition of $X$ into a finite number of disjoint measurable sets $X = \bigcup_{i=1}^k A_i$ and reals $\alpha_1, \ldots, \alpha_k$ such that

\[
s = \sum_{i=1}^k \alpha_k 1_{A_k}
\]

and we have for all $i$,

\[
\mathcal{R} \cap A_i \times \mathbb{R} = \{(x, t) \in A_i \times \mathbb{R} | 0 \leq t < s(x)\} \\
= \{(x, t) \in A_i \times \mathbb{R} | 0 \leq t < \alpha_i\} \\
= A_i \times [0, \alpha_i)
\]

Therefore we have the partition

\[
\mathcal{R} = \bigcup_{i=1}^k A_i \times [0, \alpha_i)
\]

where each element of the partition is measurable. Using additivity of $\mu \times m$,

\[
(\mu \times m)(\mathcal{R}) = \sum_{i=1}^k (\mu \times m)(A_i \times [0, \alpha_i)) \\
= \sum_{i=1}^k \mu(A_i) m([0, \alpha_i)) \\
= \sum_{i=1}^k \alpha_i \mu(A_i) \\
= \int_X s d\mu
\]
This proves the result for simple functions.

Now consider the general case. Let \( f : X \to [0, \infty) \) be measurable, and let \((s_n)_{n \in \mathbb{N}}\) be a sequence of non-negative simple functions that converges to \( f \), and such that for all \( n \), \( 0 \leq s_n \leq s_{n+1} \leq f \). For all \( n \), let \( R_n \) be the region under the graph of \( s_n \). Then we have for all \( n \), if \((x,t) \in R_n \), then \( 0 \leq t < s_n(x) \), and since \( s_n(x) \leq s_{n+1}(x) \), we also have \( 0 \leq t < s_{n+1}(x) \), i.e. \( R_n \subseteq R_{n+1} \). We also have

\[
R = \bigcup_{n \geq 1} R_n
\]

indeed, if \((x,t) \in R_n \) for some \( n \), then \( 0 \leq t < s_n(x) \), but \( s_n(x) \leq f(x) \), thus \((x,t) \in R \). Conversely, if \((x,t) \in R \), then \( 0 \leq t < f(x) \), and since \( s_n(x) \to f(x) \), there exists \( n \) such that \( 0 \leq t < s_n(x) \leq f(x) \), i.e. \((x,t) \in R_n \).

Finally, we have by the monotone converges theorem and \( \sigma \)-additivity of \( \mu \times m \)

\[
\int_X f \, d\mu = \lim_{n \to \infty} \int_X s_n \, d\mu
= \lim_{n \to \infty} (\mu \times m)R_n
= (\mu \times m) \bigcup_{n \in \mathbb{N}} R_n
= (\mu \times m)R
\]

3. \( G \) is measurable: for all \( n \geq 1 \) let \( G_n = \{(x,t) \in X \mid f(x) - 1/n < t < f(x) + 1/n \} \). Then we have

\[
(x,t) \in G \iff f(x) = t
\]

\[
\iff \forall n \geq 1, \ f(x) - 1/n < t < f(x) + 1/n
\]

\[
\iff (x,t) \in \bigcap_{n=1}^{\infty} G_n
\]

and we have for all \( n \)

\[
(x,t) \in G_n \iff f(x) - 1/n < t < f(x) + 1/n
\]

\[
\iff \exists p,q \in \mathbb{Q} : f(x) - 1/n < p < t < q < f(x) + 1/n
\]

\[
\iff \exists p,q \in \mathbb{Q} : f(x) \in (q - 1/n,p + 1/n) \text{ and } t \in (p,q)
\]

\[
\iff (x,t) \in \bigcup_{p,q \in \mathbb{Q}} \left( f^{-1}((q - 1/n,p + 1/n)) \times (q,p) \right)
\]

therefore \( G_n \) is the union of countably many measurable sets, and is measurable. It follows that \( G \) is measurable.

Finally, by the Fubini theorem, we have

\[
(\mu \times m)(G) = \int_{X \times \mathbb{R}} 1_G(x,t) \, d(\mu \times m)
= \int_X \left( \int_{\mathbb{R}} 1_G(x,t) \, dm(t) \right) \, d\mu(x)
\]

but for all \( x \in X \), \( 1_G(x,t) = 1_{\{f(x)\}}(t) \), thus \( \int_{\mathbb{R}} 1_G(x,t) \, dm(t) = \int_{\mathbb{R}} 1_{\{f(x)\}} \, dm(t) = m(\{f(x)\}) = 0 \).

Therefore

\[
(\mu \times m)(G) = \int_X 0 \, d\mu(x) = 0
\]