MATH 202A - Problem Set 12

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(12.1) For $x \in \mathbb{R}$, let $P_x = \{(p,q) \in \mathbb{Z} \times \mathbb{N} | p \wedge q \text{ and } |x - \frac{p}{q}| \leq \frac{1}{q^r}\}$, were $r > 2$. Let $E = \{x \in \mathbb{R} | P_x \text{ is an infinite set}\}$. Then we have E is a Borel set, and $m(E) = 0$.

proof For all $x \in \mathbb{R}$, let

$$Q_x = \{q \in \mathbb{N} | \exists p \in \mathbb{Z} : p \wedge q = 1 \text{ and } |x - \frac{p}{q}| \leq \frac{1}{q^r}\}$$

be the projection of P_x on the second coordinate. We first observe that P_x is infinite if and only if Q_x is infinite. Indeed

- Q_x is infinite $\Rightarrow$ P_x is infinite: this follows from the fact that the projection

$$\begin{aligned} P_x &\to Q_x \\ (p,q) &\mapsto q \end{aligned}$$

  is a surjection.

- Q_x is finite $\Rightarrow$ P_x is finite: for all $q \in Q_x$, there are at most three integers $p \in \mathbb{Z}$ such that $(p,q) \in P_x$ (indeed, if $|x - \frac{p_1}{q}| \leq \frac{1}{q^3}$ and $|x - \frac{p_2}{q}| \leq \frac{1}{q^3}$, then $|\frac{p_1}{q} - \frac{p_2}{q}| \leq \frac{2}{q^3}$, i.e. $|p_1 - p_2| \leq 2/q^{r-1} \leq 2$ $(q^{r-1} \geq 1)$, therefore there may be at most three distinct such integers p).

Now for all $q \in \mathbb{N}$ and for all $n \in \mathbb{N}$ let

$$X_q^n = \{x \in [-n,n] | \exists p \in \mathbb{Z} : p \wedge q = 1 \text{ and } |x - \frac{p}{q}| \leq \frac{1}{q^r}\}$$

we have $x \in X_q^n \Leftrightarrow q \in Q_x$ and $x \in [-n,n]$, therefore

$x \in E \cap [-n,n] \Leftrightarrow x \in [-n,n]$ and P_x is infinite $\Leftrightarrow x \in [-n,n]$ and Q_x is infinite $\Leftrightarrow \{q \in \mathbb{N} | x \in X_q^n\}$ is infinite

Therefore $E \cap [-n,n]$ is the set of points x that belong to X_q^n for infinitely many indices $q \in \mathbb{N}$. We observe that

$$X_q^n = \cup_{p \in \mathbb{Z}, p \wedge q = 1} \left([-n,n] \cap \left[\frac{p}{q} - \frac{1}{q^r}, \frac{p}{q} + \frac{1}{q^r} \right] \right)$$

therefore it is a Borel set, and we observe that each intersection $[-n,n] \cap [p/q - 1/q^r, p/q + 1/q^r]$ is nonempty if and only if $p/q - 1/q^r \leq n$ and $p/q + 1/q^r \geq -n$, i.e.

$$p \in [-nq - 1/q^{r-1}, nq + 1/q^{r-1}] \subseteq [-nq - 1, nq + 1]$$

therefore we can write

$$X_q^n \subseteq \cup_{p \in [-nq-1, nq+1]} \left[\frac{p}{q} - \frac{1}{q^r}, \frac{p}{q} + \frac{1}{q^r} \right]$$

thus the measure of X_q^n is $m(X_q^n) \leq (2nq+3)\frac{2}{q^r} \leq (2nq+3q)\frac{2}{q^r} = \frac{4n+6}{q^{r-1}}$ and

$$\sum_{q\in\mathbb{N}} m(X_q^n) \leq \sum_{q\in\mathbb{N}} \frac{4n+6}{q^{r-1}} < \infty$$

therefore by the Borel-Cantelli theorem, $m(E\cap[-n,n]) = 0$. Finally, since the sequence $(E\cap[-n,n])_n$ is non-decreasing, and $\cup_{n=1}^{\infty} E\cap[-n,n] = E$, we have by σ-additivity of the measure

$$m(E) = \lim_{n\to\infty} m(E\cap[-n,n]) = 0$$

(12.2) Let E be the set of all $x \in [0,1)$ such that x has decimal expansion $x = 0.a_1a_2\ldots$ in which no two consecutive digits equal 7. Then E is a Borel set, and $m(E) = 0$.

proof Let E_n be the set of points $x \in [0,1)$ that have decimal expansion $x = 0.a_1a_2\ldots a_n\ldots$ such that the $(a_1,\ldots,a_n)$ does not contain two consecutive elements equal to 7. Let $F_n = E_n^c$. We have $F_n \subseteq F_{n+1}$ (if $(a_1,\ldots,a_n)$ contains two consecutive elements equal to 7, then so does $(a_1,\ldots,a_{n+1})$), thus $E_{n+1} \subseteq E_n$ and we have

$$E = \cap_{n=1}^{\infty} E_n$$

We additionally write E_n as the disjoint union of two sets: the set of points in E that have digital expansion with the n^{th} digit equals 7 (respectively, not equal to 7)

$$E_n = E_n^7 \cup E_n^{\bar{7}}$$

where E_n^7 is the set of points $x \in E_n$ that have decimal expansion $x = 0.a_1\ldots a_n\ldots$ such that $a_n = 7$, and $E_n^{\bar{7}}$ is the set of points $x \in E_n$ that have decimal expansion $x = 0.a_1\ldots a_n\ldots$ such that $a_n \neq 7$.

Then we have the following equivalences: for all $n \geq 1$

$$x \in E_{n+1}^7 \Leftrightarrow x \in E_n^{\bar{7}} \text{ and } a_{n+1} = 7$$
$$x \in E_{n+1}^{\bar{7}} \Leftrightarrow x \in E_n \text{ and } a_{n+1} \neq 7$$

Now let

$$u_n = m(E_n^7)$$
$$v_n = m(E_n^{\bar{7}})$$

Then we have $m(E_n) = u_n + v_n$, and we have

$$u_{n+1} = \frac{1}{10} v_n$$
$$v_{n+1} = \frac{9}{10}(u_n + v_n)$$

with initial terms $u_1 = \frac{1}{10}$ and $v_1 = \frac{9}{10}$. Then we can write

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ (1-\alpha) & (1-\alpha) \end{pmatrix}^{n-1} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$$

with $\alpha = \frac{1}{10}$. Therefore we have $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = 0$.

Finally, since $E_{n+1} \subseteq E_n$ and $m(E_0) = 1$ is finite, we have by σ-additivity

$$\begin{aligned} m(E) &= m(\cap_{n=1}^{\infty} E_n) \\ &= \lim_{n\to\infty} m(E_n) \\ &= \lim_{n\to\infty} u_n + v_n \\ &= 0 \end{aligned}$$

(12.3) Let $(X, \mathcal{A}, \mu)$ be a measure space, with $\mu(X) = 1$. A family of measurable sets $\{A_n\}$ is said to be mutually independent if for all finite N, and all $\{B_k\}_{k\in\{1,\dots,N\}}$ with $B_k \in \{A_k, X \setminus A_k\}$ satisfies

$$\mu(\cap_{k=1}^{N} B_k) = \prod_{k=1}^{N} \mu(B_k)$$

If $\{A_n\}$ is a family of mutually independent sets, and $\sum_n \mu(A_n) = \infty$, then almost every $x \in X$ belongs to A_n for infinitely many indices n.

proof Let $E = \{x \in X | x \in A_n \text{ for infinitely many } n\}$. Then we have $x \in E$ if and only if $\forall N \in \mathbb{N}$, $\exists n \geq N$ such that $x \in A_n$, therefore

$$E = \cap_{N\in\mathbb{N}} \cup_{n\geq N} A_n$$

and it follows that E is measurable. We have $E^c = \cup_{N\in\mathbb{N}} \cap_{n\geq N} A_n^c$. Let $N \in \mathbb{N}$, the sequence of sets $(\cap_{n=N}^{N+k} A_n^c)_{k\in\mathbb{N}}$ is decreasing, thus by σ-additivity of μ, and since the space has finite measure, we have

$$\mu(\cap_{k\in\mathbb{N}}(\cap_{n=N}^{N+k} A_n^c)) = \lim_{k\to\infty} \mu(\cap_{n=N}^{N+k} A_n^c)$$

for each $k \in \mathbb{N}$, by mutual independence, we have

$$\begin{aligned}\mu(\cap_{n=N}^{N+k} A_n^c) &= \prod_{n=N}^{N+k} \mu(A_n^c)\\ &= \prod_{n=N}^{N+k} (\mu(X) - \mu(A_n))\\ &= \prod_{n=N}^{N+k} (1 - \mu(A_n))\end{aligned}$$

Where the sequence $(\prod_{n=N}^{N+k}(1 - \mu(A_n)))_k$ converges to 0: to prove this, let $P_k = \prod_{n=N}^{N+k}(1 - \mu(A_n))$. We consider two cases:

1. if there exists $n_0 \geq N$ such that $\mu(A_n) = 1$, then for all $k \geq n_0 - N$, $P_k = 0$, and the result follows.
2. if $\mu(A_n) < 1$ for all $n \geq N$, then consider

$$\begin{aligned}\ln P_k &= \sum_{n=N}^{N+k} \ln(1 - \mu(A_n))\\ &\leq \sum_{n=N}^{N+k} -\mu(A_n)\end{aligned}$$

using the fact that for all $x \in [0, 1)$, $\ln(1-x) \leq -x$. By assumption, we have $\lim_{k\to\infty} \sum_{n=N}^{N+k} \mu(A) = \infty$, therefore

$$\lim_{k\to\infty} \ln P_k = -\infty$$

therefore (P_k) converges to 0 (for all $\epsilon > 0$, there exists K such that for all $k \geq K$, $\ln P_k \leq \ln \epsilon$, thus for all $k \geq K$, $P_k \leq \epsilon$).

Therefore

$$\mu(\cap_{n=N}^{\infty} A_n^c) = \mu(\cap_{k\in\mathbb{N}}(\cap_{n=N}^{N+k} A_n^c)) = \lim_{k\to\infty} \prod_{n=N}^{N+k} (1 - \mu(A_n)) = 0$$

To conclude, we have by σ-additivity of μ

$$\mu(E^c) = \mu(\cup_{N\in\mathbb{N}}(\cap_{n=N}^{\infty} A_n^c)) \\ \leq \sum_{N\in\mathbb{N}} \mu(\cap_{n=N}^{\infty} A_n^c) = 0$$

therefore $m(E^c) = 0$, which proves that almost every x is in E.

(12.4) Egoroff's theorem: Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Suppose that (f_n) is a sequence of measurable functions $f_n : X \to \mathbb{R}$, that (f_n) converges to $f : X \to \mathbb{R}$ almost everywhere, where f is a measurable function. Then for all $\epsilon > 0$, there exists $E \in \mathcal{A}$ such that $\mu(E) < \epsilon$ and (f_n) converges to f uniformly on $X \setminus E$.

proof We first show that for all $\eta > 0$, there exists $F_\eta \in \mathcal{A}$ and $N_\eta \in \mathbb{N}$ such that $\mu(F) \leq \eta$ and for all $n \geq N_\eta$, for all $x \in F_\eta^c$, $|f_n(x) - f(x)| \leq \eta$. Let $\eta > 0$ be fixed, and consider the sets

$$G_{N,\eta} = \{x \in X | \forall n \geq N, |f_n(x) - f(x)| \leq \eta\} \\ F_{N,\eta} = G_{N,\eta}^c$$

where $\cup_{N=1}^{\infty} G_{N,\eta} = X$ since for all $x \in X$, $\lim_n f_n(x) = f(x)$ thus there exists N such that for all $n \geq N$ $|f_n(x) - f(x)| \leq \eta$, i.e. $x \in G_{n,\eta}$.

We have for all $N \in \mathbb{N}$, $G_{N,\eta}$ is the inverse image of the interval $[0, \eta]$ by the measurable function $|f_n - f|$, thus $G_{N,\eta}$ is measurable and so is $F_{N,\eta}$. And we have $G_{N,\eta} \subseteq G_{N+1,\eta}$, thus $F_{N+1,\eta} \subseteq F_{N,\eta}$. Therefore $(F_{N,\eta})_N$ is a non-increasing sequence of subsets, with intersection $\cap_{N=1}^{\infty} F_{N,\eta} = (\cup_{N=1}^{\infty} G_{N,\eta})^c = X^c = \emptyset$. by σ-additivity of m,

$$\lim_{n\to\infty} m(F_{N,\eta}) = m(\emptyset) = 0$$

thus there exists $N_\eta \in \mathbb{N}$ such that $m(F_{N_\eta,\eta}) \leq \eta$, and on $F_{N_\eta,\eta}^c = G_{N_\eta,\eta}$, we have $|f_n - f| \leq \eta$. Let F_η denote $F_{N_\eta,\eta}$ (to simplify notation).

Now fix $\epsilon > 0$, and consider the sequence $\eta_n = \epsilon/2^n$. Consider the union

$$F = \cup_{n=1}^{\infty} F_{\eta_n}$$

then we have

- F is measurable as the countable union of measurable sets.
- the measure of F is

$$m(F) = m(\cup_{n=1}^{\infty} F_{\eta_n}) \leq \sum_{n=1}^{\infty} m(F_{\eta_n}) \\ \leq \sum_{n=1}^{\infty} \eta_n = \epsilon \sum_{n=1}^{\infty} 1/2^n = \epsilon$$

- on the complement of F, f_n converges to f uniformly: indeed, for all $\eta > 0$, since $\lim_n \eta_n = 0$, there exists n_0 such that for all $\eta_{n_0} \leq \eta$, and we have for all $F_{\eta_{n_0}} \subseteq F$, i.e. $F^c \subseteq F_{\eta_{n_0}}$, therefore for all $x \in F^c$, $x \in F_{\eta_{n_0}}$, thus for all $n \geq N_{\eta_{n_0}}$, $|f_n(x) - f(x)| \leq \eta$. This proves that (f_n) converges to f uniformly on F^c.

(12.5) Let $(X, \mathcal{A}, \mu)$ be a measure space, such that $\mu(X) < \infty$. Then the dominated convergence theorem follows from Egoroff's theorem:

Let $E \in \mathcal{A}$, and let (f_n) be a sequence of measurable functions that converges pointwise to f, and such that there exists an integrable g with $|f_n(x)| \leq g(x)$ for all $x \in X$. Then f is integrable, and $\left(\int_E f_n d\mu\right)_n$ converges to $\int_E f d\mu$.

proof

Lemma 1 *If $f : X \to \mathbb{R}$ is integrable, then for all $\epsilon > 0$, there exists $\eta > 0$ such that if $F \in \mathcal{A}$ has measure $\mu(F) \leq \eta$, then $\int_F h d\mu \leq \epsilon$.*

First, we have f is measurable as the pointwise limit of the sequence of measurable functions (f_n). Then we have for all x, $|f(x)| = \lim_{n\to\infty} |f_n(x)| \leq g(x)$, thus

$$\int_X |f| d\mu \leq \int_X g d\mu < \infty$$

therefore f is integrable.

Since g is integrable, by Lemma 1, there exists $\eta > 0$ such that for every measurable set $F \in \mathcal{A}$ with measure $m(F) \leq \eta$, $\int_F g d\mu \leq \epsilon/3$.

Applying Egoroff's theorem to the sequence of functions (f_n) that converges pointwise to f, there exists a measurable set F such that $\mu(F) \leq \eta$ and (f_n) converges uniformly to f on F^c. Consider the constant $c = \frac{\epsilon}{3\mu(X)}$ (assuming that $\mu(X) > 0$, otherwise the result is trivial), then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, for all $x \in F^c$, $|f_n(x) - f(x)| \leq c$. Thus for all $n \geq N$

$$\begin{aligned} |\int_E f_n d\mu - \int_E f d\mu| &= |\int_E (f_n - f) d\mu| \\ &\leq \int_E |f_n - f| d\mu \\ &= \int_{E\cap F} |f_n - f| d\mu + \int_{E\cap F^c} |f_n - f| d\mu \end{aligned}$$

To bound the first term in the sum, we have

$$\int_{E\cap F} |f_n| + |f| d\mu \leq 2 \int_{E\cap F} g d\mu \leq 2\frac{\epsilon}{3}$$

since $\mu(E \cap F) \leq \mu(F) \leq \eta$. To bound the second term, we have

$$\int_{E\cap F^c} |f_n - f| d\mu \leq \int_{E\cap F^c} c d\mu \leq c\mu(E \cap F^c) \leq c\mu(X) = \frac{\epsilon}{3}$$

Adding the two terms, we obtain

$$|\int_E f_n d\mu - \int_E f d\mu| \leq \epsilon$$

which proves that $(\in_E f_n d\mu)$ converges to $\int_E f d\mu$.

(12.6) Let $(X, \mathcal{A}, \mu)$ be a measure space such that $\mu(X)$ is finite. Let $f_n : X \to [0, \infty)$ be a measurable function, and suppose that (f_n) converges to f almost everywhere. Then

1. if $\int_X f d\mu < \infty$ and $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$, then $\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu$ for every measurable set E.
2. this is not true in general if $\int_X f d\mu$ is not finite.
3. this is not true in general if $f_n : X \to \mathbb{R}$ instead of $\mathbb{R}_+$, and assumed to be integrable.
4. under the assumptions of 1, we additionally have $\int_X |f_n - f| d\mu \to 0$

proof Let $A = \{x \in X | f_n(x) \to f(x)\}$. Then A is measurable, and by assumption, $m(A^c) = 0$.

1. Let $E \in \mathcal{A}$ be a measurable set. Since (f_n) is a sequence of nonegative measurable functions, we have by Fatou's Lemma
$$\int_E \liminf_n f_n d\mu \leq \liminf_n \int_E f_n$$
but $\int_E \liminf f_n d\mu = \int_E f d\mu$ since the difference
$$\int_E (\liminf_n f_n - f) d\mu = \int_{A\cap E} (\liminf_n f_n - f) d\mu + \int_{A^c \cap E} (\liminf_n f_n - f) d\mu = 0 + 0$$
where the first term is zero since for all $x \in A$, $\liminf_n f_n(x) = \lim_n f_n(x) = f(x)$, and the second term is zero since $m(A^c \cap E) \leq m(A^c) = 0$. Therefore we have
$$\int_E f d\mu \leq \liminf_n \int_E f_n$$
Similarly, we have
$$\int_{E^c} f d\mu \leq \liminf_n \int_{E^c} f_n \tag{1}$$
Now we have for all $k \in \mathbb{N}$
$$\begin{aligned} \sup_{n\geq k} \int_E f_n d\mu &= \sup_{n\geq k} \left(\int_X f_n d\mu - \int_{E^c} f_n d\mu \right) && \text{since } X = E \cup E^c \text{ disjointly} \\ &\leq \sup_{n\geq k} \int_X f_n d\mu + \sup_{n\geq k} - \int_{E^c} f_n d\mu \\ &= \sup_{n\geq k} \int_X f_n d\mu - \inf_{n\geq k} \int_{E^c} f_n d\mu \end{aligned}$$
since both sequences converge, so does the sum of sequences, and we have
$$\begin{aligned} \limsup_n \int_E f_n d\mu &\leq \limsup_n \int_X f_n d\mu - \liminf_n \int_{E^c} f_n d\mu \\ &= \limsup_n \int_X f_n d\mu - \liminf_n \int_{E^c} f_n d\mu \\ &\leq \int_X f d\mu - \liminf_n \int_{E^c} f_n d\mu && \text{since } \left(\int_X f_n d\mu \right)_n \to \int_X f d\mu \\ &\leq \int_X f d\mu - \int_{E^c} \liminf_n f_n d\mu && \text{by (1)} \\ &= \int_E f d\mu + \int_{E^c} f d\mu - \int_{E^c} \liminf_n f_n d\mu \\ &= \int_E f d\mu \end{aligned}$$

Therefore we have

$$\limsup_n \int_E f_n d\mu \leq \int_E f d\mu \leq \liminf_n \int_E f_n d\mu \leq \limsup_n \int_E f_n d\mu$$

which proves that all terms are equal, thus $\left(\int_E f_n d\mu\right)_n$ converges, and

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu$$

2. Counterexample with $\int_X f d\mu = \infty$: consider the sequence of functions

$$\begin{aligned} f_n : \mathbb{R}_+ &\to \mathbb{R}_+ \\ x \mapsto f_n(x) &= \begin{cases} n(1-nx) & \text{if } 0 \leq x \leq 1/n \\ 0 & \text{if } 1/n < x \leq 1 \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

then we have

- (f_n) converges pointwise to

$$\begin{aligned} f : \mathbb{R}_+ &\to \mathbb{R}_+ \\ x \mapsto f(x) &= \begin{cases} +\infty & \text{if } x = 0 \\ 0 & \text{if } 0 < x \leq 1 \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

- for all $n \in \mathbb{N}$, $\int_{\mathbb{R}_+} f_n dm = \int_{\mathbb{R}_+} f dm = \infty$.
- however, $\int_{[0,1]} f_n dm = 1/2$ for all n, while $\int_{[0,1]} f dm = 0$.

3. Counterexample with $f_n : X \to \mathbb{R}$ instead of $\mathbb{R}_+$, and integrable.

$$\begin{aligned} f_n : \mathbb{R} &\to \mathbb{R} \\ x \mapsto f_n(x) &= \begin{cases} x/n^2 & \text{if } -n \leq x \leq n \\ 0 & \text{if } 1/n < x \leq 1 \end{cases} \end{aligned}$$

then we have

- for all n, f_n is integrable and $\int_{\mathbb{R}} |f_n| dm = 1$
- (f_n) converges pointwise to the function f identically zero on $\mathbb{R}$
- for all $n \in \mathbb{N}$, $\int_{\mathbb{R}} f_n dm = \int_{\mathbb{R}} f dm = 0$.
- however, $\int_{\mathbb{R}_+} f_n dm = 1/2$ for all n, while $\int_{\mathbb{R}_+} f dm = 0$.

4. Let B be the measurable set of points where f is nonzero,

$$B = \{x \in X | f(x) > 0\}$$

We have

$$\int_X |f - f_n| d\mu = \int_{B^c} f_n d\mu + \int_B |f - f_n| d\mu$$

In order to apply Egoroff's theorem, we need a measure space with finite measure. Define the measure ν by

$$\begin{aligned} \nu : \mathcal{A} &\to \mathbb{R}_+ \\ E &\mapsto \nu(E) = \int_E f d\mu \end{aligned}$$

since f is measurable and non-negative, ν is a measure, and we have

$$\nu(X) = \int_X f d\mu < \infty$$

We assume $\nu(X) > 0$, otherwise f is zero almost everywhere, and the result becomes trivial. Now consider the functions

$$\begin{aligned} g_n : X &\to \mathbb{R}_+ \\ x &\mapsto g_n(x) = \begin{cases} \frac{f_n(x)}{f(x)} & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We have

- g_n is a measurable function
- g_n converges pointwise to 1_B on A (indeed, if $x \in A \cap B^c$, then $g_n(x) = 1_B(x) = 0$, and if $x \in A \cap B$, then $(f_n(x)) \to f(x) > 0$, thus $(g_n(x)) \to 1 = 1_B(x)$), and $\nu(A^c) = \int_{A^c} f d\mu = 0$ since $\mu(A^c) = 0$. Therefore $(g_n) \to 1_B$ almost everywhere.

We can now apply Egoroff's theorem on (g_n). Let $\epsilon > 0$. By Egoroff's theorem, there exists $F \in \mathcal{A}$ such that $\nu(F) \leq \epsilon/4$ (i.e. $\int_F f d\mu \leq \epsilon/4$) and (g_n) converges uniformly to 1_B on F^c. Consider the constant $c = \frac{\epsilon}{4\nu(X)}$ (note that this is well defined since we assumed $\nu(X) > 0$). Then there exists N_1 such that for all $n \geq N_1$, for all $x \in F^c$, $|g_n(X) - 1_B(x)| \leq c$. In particular,

$$\forall x \in B \cap F^c,\ |f_n(x) - f(x)| \leq c f(x)$$

We then have

$$\begin{aligned} \int_X |f - f_n| d\mu &= \int_{B^c} f_n + \int_{B \cap F} |f - f_n| d\mu + \int_{B \cap F^c} |f - f_n| d\mu \\ &\leq \int_{B^c} f_n + \int_{B \cap F} f d\mu + \int_{B \cap F} f_n d\mu + \int_{B \cap F^c} |f - f_n| d\mu \end{aligned}$$

since $\int_E f_n d\mu$ converges to $\int_E f d\mu$ for all $E \in \mathcal{A}$ (by 1), there exists N_2 such that for all $n \geq N_2$, $\int_{B^c} f_n \leq \epsilon/4 + \int_{B^c} f d\mu = \epsilon/4 + 0$ and $\int_{B \cap F} f_n d\mu \leq \epsilon/4 + \int_{B \cap F} f d\mu \leq \epsilon/4 + \epsilon/4$. Therefore we have for all $n \geq \max(N_1, N_2)$,

$$\begin{aligned} \int_X |f - f_n| d\mu &\leq 3\epsilon/4 + \int_{B \cap F^c} |f - f_n| d\mu \\ &\leq 3\epsilon/4 + c \int_{B \cap F^c} f d\mu \\ &\leq 3\epsilon/4 + c \int_X f d\mu \\ &= 3\epsilon/4 + \epsilon/4 \end{aligned}$$

which proves $\int_X |f - f_n| d\mu$ converges to 0.

(12.7) $\lim_{n\to\infty}\int_0^n(1-\frac{x}{n})^n e^{x/2}dx = 2.$

proof For all $n \in \mathbb{N}$, let

$$\begin{aligned} f_n : \mathbb{R}_+ &\to \mathbb{R} \\ x &\mapsto f(x) = (1-x/n)^n e^{x/2} 1_{[0,n]} \end{aligned}$$

We seek to compute $\lim_{n\to\infty}\int_{\mathbb{R}_+} f_n(x)dx$. We first observe that for all $n \in \mathbb{N}$, $f_n(x) \le f_{n+1}(x)$.[1] Let $x \ge 0$. Then $\lim_{n\to\infty} f_n(x) = e^{-x/2}$, since for all $n > x$, $f_n(x) = (1-x/n)^n e^{x/2} = \exp(n\log(1-x/n) + x/2) = \exp(n(-x/n + o(n)) + x/2) = \exp(-x/2 + o(1))$. Therefore for all $n \in \mathbb{N}$, for all $x \ge 0$

$$0 \le f_n(x) \le e^{-x/2}$$

where $x \mapsto e^{-x/2}$ is integrable on $\mathbb{R}_+$. By the dominated convergence theorem, $\lim_{n\to\infty}\int_{\mathbb{R}_+} f_n(x)dx$ exists and

$$\begin{aligned} \lim_{n\to\infty}\int_0^n (1-\frac{x}{n})^n e^{x/2}dx &= \lim_{n\to\infty}\int_{\mathbb{R}_+} f_n(x)dx \\ &= \int_{\mathbb{R}_+} e^{-x/2}dx \\ &= 2 \end{aligned}$$

[1] $f_n(x) \le f_{n+1}(x)$: this is true for $x > n$. To prove this for $x \in [0, n)$, consider

$$\begin{aligned} h : [0,n) &\to \mathbb{R} \\ x \mapsto h(x) \quad &= \log(f_{n+1}(x)/f_n(x)) \\ &= (n+1)\log(1-\frac{x}{n}) - n\log(1-\frac{x}{n}) \end{aligned}$$

This is well defined since $f_{n+1}(x)/f_n(x) > 0$. Then we have for all $x \in [0, n)$

$$h'(x) = -\frac{1}{\log(1-\frac{x}{n+1})} + \frac{1}{\log(1-\frac{x}{n})} > 0$$

therefore h is increasing, and for all $x \in [0, n)$, $h(x) \ge h(0) = 0$, therefore $f_{n+1}(x)/f_n(x) \ge 1$.