(12.1) For \( x \in \mathbb{R} \), let \( P_x = \{(p,q) \in \mathbb{Z} \times \mathbb{N} | p \land q \text{ and } |x - \frac{p}{q}| \leq \frac{1}{q^r}\} \), were \( r > 2 \). Let \( E = \{x \in \mathbb{R} | P_x \text{ is an infinite set}\} \). Then we have \( E \) is a Borel set, and \( m(E) = 0 \).

**proof** For all \( x \in \mathbb{R} \), let

\[
Q_x = \{q \in \mathbb{N} | \exists p \in \mathbb{Z} : p \land q = 1 \text{ and } |x - \frac{p}{q}| \leq \frac{1}{q^r}\}
\]

be the projection of \( P_x \) on the second coordinate. We first observe that \( P_x \) is infinite if and only if \( Q_x \) is infinite. Indeed

- \( Q_x \) is infinite \( \Rightarrow \) \( P_x \) is infinite: this follows from the fact that the projection \( P_x \to Q_x \)

\[
(p,q) \mapsto q
\]

is a surjection.

- \( Q_x \) is finite \( \Rightarrow \) \( P_x \) is finite: for all \( q \in Q_x \), there are at most three integers \( p \in \mathbb{Z} \) such that \((p,q) \in P_x\) (indeed, if \( |x - \frac{p_1}{q}| \leq \frac{1}{q^r} \) and \( |x - \frac{p_2}{q}| \leq \frac{1}{q^r} \), then \( |\frac{p_1}{q} - \frac{p_2}{q}| \leq \frac{2}{q^r} \), i.e. \( |p_1 - p_2| \leq 2/q^r - 1 \leq 2 (q^r - 1) \), therefore there may be at most three distinct such integers \( p \)).

Now for all \( q \in \mathbb{N} \) and for all \( n \in \mathbb{N} \) let

\[
X_q^n = \{x \in [-n,n] | \exists p \in \mathbb{Z} : p \land q = 1 \text{ and } |x - \frac{p}{q}| \leq \frac{1}{q^r}\}
\]

we have \( x \in X_q^n \iff q \in Q_x \) and \( x \in [-n,n] \), therefore

\( x \in E \cap [-n,n] \iff x \in [-n,n] \) and \( P_x \) is infinite \( \iff x \in [-n,n] \) and \( Q_x \) is infinite \( \iff \{q \in \mathbb{N} | x \in X_q^n\} \) is infinite.

Therefore \( E \cap [-n,n] \) is the set of points \( x \) that belong to \( X_q^n \) for infinitely many indices \( q \in \mathbb{N} \). We observe that

\[
X_q^n = \bigcup_{p \in \mathbb{Z}, p \land q = 1} \left( \left([-n,n] \cap \left[\frac{p}{q} - \frac{1}{q^r}, \frac{p}{q} + \frac{1}{q^r}\right]\right) \right)
\]

therefore it is a Borel set, and we observe that each intersection \( [-n,n] \cap [p/q - 1/q^r, p/q + 1/q^r] \) is nonempty if and only if \( p/q - 1/q^r \leq n \) and \( p/q + 1/q^r \geq -n \), i.e.

\[
p \in [-nq - 1/q^r - 1, nq + 1/q^r - 1] \subseteq [-nq - 1, nq + 1]
\]

therefore we can write

\[
X_q^n \subseteq \bigcup_{p \in [-nq - 1, nq + 1]} \left[\frac{p}{q} - \frac{1}{q^r}, \frac{p}{q} + \frac{1}{q^r}\right]
\]
thus the measure of $X^m$ is $m(X^m) \leq (2nq + 3) \frac{2}{q'} \leq (2nq + 3q) \frac{2}{q'} = \frac{4n + 6}{q'r-1}$ and

$$\sum_{q \in \mathbb{N}} m(X^m) \leq \sum_{q \in \mathbb{N}} \frac{4n + 6}{q'r-1} < \infty$$

therefore by the Borel-Cantelli theorem, $m(E \cap [-n, n]) = 0$. Finally, since the sequence $(E \cap [-n, n])_n$ is non-decreasing, and $\bigcup_{n=1}^\infty E \cap [-n, n] = E$, we have by $\sigma$-additivity of the measure

$$m(E) = \lim_{n \to \infty} m(E \cap [-n, n]) = 0$$

(12.2) Let $E$ be the set of all $x \in [0, 1)$ such that $x$ has decimal expansion $x = 0.a_1a_2\ldots$ in which no two consecutive digits equal 7. Then $E$ is a Borel set, and $m(E) = 0$.

**proof** Let $E_n$ be the set of points $x \in [0, 1)$ that have decimal expansion $x = 0.a_1a_2\ldots a_n\ldots$ such that the $(a_1, \ldots, a_n)$ does not contain two consecutive elements equal to 7. Let $F_n = E_n^c$. We have $F_n \subseteq F_{n+1}$ (if $(a_1, \ldots, a_n)$ contains two consecutive elements equal to 7, then so does $(a_1, \ldots, a_{n+1})$), thus $E_{n+1} \subseteq E_n$ and we have

$$E = \cap_{n=1}^\infty E_n$$

We additionally write $E_n$ as the disjoint union of two sets: the set of points in $E$ that have digital expansion with the $n$th digit equals 7 (respectively, not equal to 7)

$$E_n = E_n^7 \cup E_n^{5/6}$$

where $E_n^7$ is the set of points $x \in E_n$ that have decimal expansion $x = 0.a_1\ldots a_n\ldots$ such that $a_n = 7$, and $E_n^{5/6}$ is the set of points $x \in E_n$ that have decimal expansion $x = 0.a_1\ldots a_n\ldots$ such that $a_n \neq 7$.

Then we have the following equivalences: for all $n \geq 1$

$$x \in E_n^{7} \Leftrightarrow x \in E_n^7 \text{ and } a_{n+1} = 7$$

$$x \in E_n^{7} \Leftrightarrow x \in E_n \text{ and } a_{n+1} \neq 7$$

Now let

$$u_n = m(E_n^7)$$

$$v_n = m(E_n^{5/6})$$

Then we have $m(E_n) = u_n + v_n$, and we have

$$u_{n+1} = \frac{1}{10} v_n$$

$$v_{n+1} = \frac{9}{10} (u_n + v_n)$$

with initial terms $u_1 = \frac{1}{10}$ and $v_1 = \frac{9}{10}$. Then we can write

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ (1 - \alpha) \end{pmatrix} \begin{pmatrix} \alpha \\ (1 - \alpha) \end{pmatrix}^{n-1} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$$

with $\alpha = \frac{1}{10}$. Therefore we have $\lim_{n \to \infty} u_n = \lim_{n \to \infty} v_n = 0$.

Finally, since $E_{n+1} \subseteq E_n$ and $m(E_0) = 1$ is finite, we have by $\sigma$-additivity

$$m(E) = m(\cap_{n=1}^\infty E_n)$$

$$= \lim_{n \to \infty} m(E_n)$$

$$= \lim_{n \to \infty} u_n + v_n$$

$$= 0$$
Let $(X, \mathcal{A}, \mu)$ be a measure space, with $\mu(X) = 1$. A family of measurable sets $\{A_n\}$ is said to be mutually independent if for all finite $N$, and all $\{B_k\}_{k \in \{1, \ldots, N\}}$ with $B_k \in \{A_k, X \setminus A_k\}$ satisfies

$$
\mu(\cap_{k=1}^{N} B_k) = \prod_{k=1}^{N} \mu(B_k)
$$

If $\{A_n\}$ is a family of mutually independent sets, and $\sum_n \mu(A_n) = \infty$, then almost every $x \in X$ belongs to $A_n$ for infinitely many indices $n$.

**proof**  Let $E = \{x \in X | x \in A_n \text{ for infinitely many } n\}$. Then we have $x \in E$ if and only if $\forall N \in \mathbb{N}, \exists n \geq N$ such that $x \in A_n$, therefore

$$
E = \cap_{N \in \mathbb{N}} \cup_{n \geq N} A_n
$$

and it follows that $E$ is measurable. We have $E^c = \cup_{N \in \mathbb{N}} \cap_{n \geq N} A_n^c$. Let $N \in \mathbb{N}$, the sequence of sets $(\cap_{n=N}^{N+k} A_n^c)_{k \in \mathbb{N}}$ is decreasing, thus by $\sigma$-additivity of $\mu$, and since the space has finite measure, we have

$$
\mu(\cap_{k \in \mathbb{N}}(\cap_{n=N}^{N+k} A_n^c)) = \lim_{k \to \infty} \mu(\cap_{n=N}^{N+k} A_n^c)
$$

for each $k \in \mathbb{N}$, by mutual independence, we have

$$
\mu(\cap_{n=N}^{N+k} A_n^c) = \prod_{n=N}^{N+k} \mu(A_n^c) = \prod_{n=N}^{N+k} (\mu(X) - \mu(A_n)) = \prod_{n=N}^{N+k} (1 - \mu(A_n))
$$

Where the sequence $(\prod_{n=N}^{N+k} (1 - \mu(A_n)))_k$ converges to 0: to prove this, let $P_k = \prod_{n=N}^{N+k} (1 - \mu(A_n))$. We consider two cases:

1. if there exists $n_0 \geq N$ such that $\mu(A_{n_0}) = 1$, then for all $k \geq n_0 - N, P_k = 0$, and the result follows.
2. if $\mu(A_n) < 1$ for all $n \geq N$, then consider

$$
\ln P_k = \sum_{n=N}^{N+k} \ln (1 - \mu(A_n)) \leq \sum_{n=N}^{N+k} -\mu(A_n)
$$

using the fact that for all $x \in [0, 1)$, $\ln(1-x) \leq -x$. By assumption, we have $\lim_{k \to \infty} \sum_{n=N}^{N+k} \mu(A) = \infty$, therefore

$$
\lim_{k \to \infty} \ln P_k = -\infty
$$

therefore $(P_k)$ converges to 0 (for all $\epsilon > 0$, there exists $K$ such that for all $k \geq K$, $\ln P_k \leq \ln \epsilon$, thus for all $k \geq K, P_k \leq \epsilon$).

Therefore

$$
\mu(\cap_{n=N}^{\infty} A_n^c) = \mu(\cap_{k \in \mathbb{N}}(\cap_{n=N}^{N+k} A_n^c)) = \lim_{k \to \infty} \prod_{n=N}^{N+k} (1 - \mu(A_n)) = 0
$$
We first show that for all $X$ uniformly measurable function. Then for all $\epsilon > 0$, there exists $E \in \mathcal{A}$ such that $\mu(E) < \epsilon$ and $(f_n)$ converges to $f$ uniformly on $X \setminus E$.

**proof** We first show that for all $\eta > 0$, there exists $F_\eta \in \mathcal{A}$ and $N_\eta \in \mathbb{N}$ such that $\mu(F) \leq \eta$ and for all $n \geq N_\eta$, for all $x \in F_\eta$, $|f_n(x) - f(x)| \leq \eta$. Let $\eta > 0$ be fixed, and consider the sets

$$
G_{N,\eta} = \{ x \in X | \forall n \geq N, |f_n(x) - f(x)| \leq \eta \}
$$

$$
F_{N,\eta} = G_{N,\eta}^c
$$

where $\bigcup_{N=1}^{\infty} G_{N,\eta} = X$ since for all $x \in X$, $\lim_{n} f_n(x) = f(x)$ thus there exists $N$ such that for all $n \geq N$ $|f_n(x) - f(x)| \leq \eta$, i.e. $x \in G_{N,\eta}$.

We have for all $N \in \mathbb{N}$, $G_{N,\eta}$ is the inverse image of the interval $[0, \eta]$ by the measurable function $|f_n - f|$, thus $G_{N,\eta}$ is measurable and so is $F_{N,\eta}$. And we have $G_{N,\eta} \subseteq G_{N+1,\eta}$, thus $F_{N+1,\eta} \subseteq F_{N,\eta}$. Therefore $(F_{N,\eta})_{N}$ is a non-increasing sequence of subsets, with intersection $\bigcap_{N=1}^{\infty} F_{N,\eta} = (\bigcup_{N=1}^{\infty} G_{N,\eta})^c = X^c = \emptyset$. by $\sigma$-additivity of $m$,

$$
\lim_{n \to \infty} m(F_{N,\eta}) = m(\emptyset) = 0
$$

thus there exists $N_\eta \in \mathbb{N}$ such that $m(F_{N_\eta,\eta}) \leq \eta$, and on $F_{N_\eta,\eta}^c = G_{N_\eta,\eta}$, we have $|f_n - f| \leq \eta$. Let $F_\eta$ denote $F_{N_\eta,\eta}$ (to simplify notation).

Now fix $\epsilon > 0$, and consider the sequence $\eta_n = \epsilon/2^n$. Consider the union

$$
F = \bigcup_{n=1}^{\infty} F_{\eta_n}
$$

then we have

- $F$ is measurable as the countable union of measurable sets.

- the measure of $F$ is

$$
m(F) = m(\bigcup_{n=1}^{\infty} F_{\eta_n}) \leq \sum_{n=1}^{\infty} m(F_{\eta_n})
$$

$$
\leq \sum_{n=1}^{\infty} \eta_n = \epsilon \sum_{n=1}^{\infty} 1/2^n = \epsilon
$$

- on the complement of $F$, $f_n$ converges to $f$ uniformly: indeed, for all $\eta > 0$, since $\lim_{n} \eta_n = 0$, there exists $n_0$ such that for all $\eta_{n_0} \leq \eta$, and we have for all $F_{n_0} \subseteq F$, i.e. $F^c \subseteq F_{n_0}$, therefore for all $x \in F^c$, $x \in F_{n_0}$, thus for all $n \geq N_{n_0}$, $|f_n(x) - f(x)| \leq \eta$. This proves that $(f_n)$ converges to $f$ uniformly on $F^c$.
Let \((X, \mathcal{A}, \mu)\) be a measure space, such that \(\mu(X) < \infty\). Then the dominated convergence theorem follows from Egoroff’s theorem:

Let \(E \in \mathcal{A}\), and let \((f_n)\) be a sequence of measurable functions that converges pointwise to \(f\), and such that there exists an integrable \(g\) with \(|f_n(x)| \leq g(x)\) for all \(x \in X\). Then \(f\) is integrable, and \(\left(\int_E f_n d\mu\right)_n\) converges to \(\int_E f d\mu\).

**proof**

**Lemma 1** If \(f : X \to \mathbb{R}\) is integrable, then for all \(\varepsilon > 0\), there exists \(\eta > 0\) such that if \(F \in \mathcal{A}\) has measure \(\mu(F) \leq \eta\), then \(\int_F h d\mu \leq \varepsilon/3\).

First, we have \(f\) is measurable as the pointwise limit of the sequence of measurable functions \((f_n)\). Then we have for all \(x\), \(|f(x)| = \lim_{n \to \infty} |f_n(x)| \leq g(x)\), thus

\[
\int_X |f| d\mu \leq \int_X g d\mu < \infty
\]

therefore \(f\) is integrable.

Since \(g\) is integrable, by Lemma 1, there exists \(\eta > 0\) such that for every measurable set \(F \in \mathcal{A}\) with measure \(m(F) \leq \eta\), \(\int_F g d\mu \leq \varepsilon/3\).

Applying Egoroff’s theorem to the sequence of functions \((f_n)\) that converges pointwise to \(f\), there exists a measurable set \(F\) such that \(\mu(F) \leq \eta\) and \((f_n)\) converges uniformly to \(f\) on \(F^c\). Consider the constant \(c = \frac{\varepsilon}{3\mu(X)}\) (assuming that \(\mu(X) > 0\), otherwise the result is trivial), then there exists \(N \in \mathbb{N}\) such that for all \(n \geq N\), for all \(x \in F^c\), \(|f_n(x) - f(x)| \leq c\). Thus for all \(n \geq N\)

\[
|\int_E f_n d\mu - \int_E f d\mu| = |\int_E (f_n - f) d\mu|
\]

\[
\leq \int_E |f_n - f| d\mu
\]

\[
= \int_{E \cap F} |f_n - f| d\mu + \int_{E \cap F^c} |f_n - f| d\mu
\]

To bound the first term in the sum, we have

\[
\int_{E \cap F} |f_n| + |f| d\mu \leq 2 \int_{E \cap F} g d\mu \leq 2 \frac{\varepsilon}{3}
\]

since \(\mu(E \cap F) \leq \mu(F) \leq \eta\). To bound the second term, we have

\[
\int_{E \cap F^c} |f_n - f| d\mu \leq \int_{E \cap F^c} c d\mu \leq c \mu(E \cap F^c) \leq c \mu(X) = \frac{\varepsilon}{3}
\]

Adding the two terms, we obtain

\[
|\int_E f_n d\mu - \int_E f d\mu| \leq \varepsilon
\]

which proves that \((\in_E f_n d\mu)\) converges to \(\int_E f d\mu\).
Let \((X, \mathcal{A}, \mu)\) be a measure space such that \(\mu(X)\) is finite. Let \(f_n : X \to [0, \infty)\) be a measurable function, and suppose that \((f_n)\) converges to \(f\) almost everywhere. Then

1. if \(\int_X f d\mu < \infty\) and \(\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu\), then \(\lim_{n \to \infty} \int_E f_n d\mu = \int_E f d\mu\) for every measurable set \(E\).

2. this is not true in general if \(\int_X f d\mu\) is not finite.

3. this is not true in general if \(f_n : X \to \mathbb{R}\) instead of \(\mathbb{R}_+\), and assumed to be integrable.

4. under the assumptions of 1, we additionally have \(\int_X |f_n - f| d\mu \to 0\).

**proof**

Let \(A = \{x \in X | f_n(x) \to f(x)\}\). Then \(A\) is measurable, and by assumption, \(m(A^c) = 0\).

1. Let \(E \in \mathcal{A}\) be a measurable set. Since \((f_n)\) is a sequence of nonegative measurable functions, we have by Fatou’s Lemma

\[
\int_E \liminf_n f_n d\mu \leq \liminf_n \int_E f_n d\mu
\]

but \(\int_E \liminf_n f_n d\mu = \int_E f d\mu\) since the difference

\[
\int_E (\liminf_n f_n - f) d\mu = \int_{A \cap E} (\liminf_n f_n - f) d\mu + \int_{A^c \cap E} (\liminf_n f_n - f) d\mu = 0 + 0
\]

where the first term is zero since for all \(x \in A\), \(\liminf_n f_n(x) = \lim_n f_n(x) = f(x)\), and the second term is zero since \(m(A^c \cap E) \leq m(A^c) = 0\). Therefore we have

\[
\int_E f d\mu \leq \liminf_n \int_E f_n d\mu
\]

Similarly, we have

\[
\int_{E^c} f d\mu \leq \liminf_n \int_{E^c} f_n d\mu
\]  \hspace{1cm} (1)

Now we have for all \(k \in \mathbb{N}\)

\[
\sup_{n \geq k} \int_E f_n d\mu = \sup_{n \geq k} \left( \int_X f_n d\mu - \int_{E^c} f_n d\mu \right) \leq \sup_{n \geq k} \int_X f_n d\mu + \sup_{n \geq k} \int_{E^c} f_n d\mu
\]

since \(X = E \cup E^c\) disjointly

\[
= \sup_{n \geq k} \int_X f_n d\mu - \inf_{n \geq k} \int_{E^c} f_n d\mu
\]

since both sequences converge, so does the sum of sequences, and we have

\[
\limsup_n \int_E f_n d\mu \leq \limsup_{n \geq k} \int_X f_n d\mu - \liminf_{n \geq k} \int_{E^c} f_n d\mu
\]

\[
\leq \int_X f d\mu - \liminf_n \int_{E^c} f_n d\mu \leq \int_X f d\mu - \int_{E^c} \liminf_n f_n d\mu
\]

by (1)

\[
\leq \int_X f d\mu + \int_{E^c} f d\mu - \int_{E^c} \liminf_n f_n d\mu
\]

\[
= \int_E f d\mu
\]
Therefore we have

\[ \limsup_n \int_E f_n d\mu \leq \int_E f d\mu \leq \liminf_n \int_E f_n d\mu \]

which proves that all terms are equal, thus \((\int_E f_n d\mu)_n\) converges, and

\[ \lim_{n \to \infty} \int_E f_n d\mu = \int_E f d\mu \]

2. Counterexample with \(\int_X f d\mu = \infty\): consider the sequence of functions

\[ f_n : \mathbb{R}_+ \to \mathbb{R}_+ \]

\[ x \mapsto f_n(x) = \begin{cases} 
  n(1 - nx) & \text{if } 0 \leq x \leq 1/n \\
  0 & \text{if } 1/n < x \leq 1 \\
  1 & \text{otherwise}
\end{cases} \]

then we have

- \((f_n)\) converges pointwise to

\[ f : \mathbb{R}_+ \to \mathbb{R}_+ \]

\[ x \mapsto f(x) = \begin{cases} 
  +\infty & \text{if } x = 0 \\
  0 & \text{if } 0 < x \leq 1 \\
  1 & \text{otherwise}
\end{cases} \]

- for all \(n \in \mathbb{N}\), \(\int_{\mathbb{R}_+} f_n dm = \int_{\mathbb{R}_+} f dm = \infty\).
- however, \(\int_{[0,1]} f_n dm = 1/2\) for all \(n\), while \(\int_{[0,1]} f dm = 0\).

3. Counterexample with \(f_n : X \to \mathbb{R}\) instead of \(\mathbb{R}_+\), and integrable.

\[ f_n : \mathbb{R} \to \mathbb{R} \]

\[ x \mapsto f_n(x) = \begin{cases} 
  x/n^2 & \text{if } -n \leq x \leq n \\
  0 & \text{if } 1/n < x \leq 1
\end{cases} \]

then we have

- for all \(n\), \(f_n\) is integrable and \(\int_{\mathbb{R}} |f_n| dm = 1\)
- \((f_n)\) converges pointwise to the function \(f\) identically zero on \(\mathbb{R}\)
- for all \(n \in \mathbb{N}\), \(\int_{\mathbb{R}} f_n dm = \int_{\mathbb{R}} f dm = 0\).
- however, \(\int_{\mathbb{R}_+} f_n dm = 1/2\) for all \(n\), while \(\int_{\mathbb{R}} f dm = 0\).

4. Let \(B\) be the measurable set of points where \(f\) is nonzero,

\[ B = \{x \in X | f(x) > 0\} \]

We have

\[ \int_X |f - f_n| d\mu = \int_{B^c} f_n d\mu + \int_B |f - f_n| d\mu \]
In order to apply Egoroff’s theorem, we need a measure space with finite measure. Define the measure \( \nu \) by

\[
\nu : \mathcal{A} \rightarrow \mathbb{R}_+
\]

\[
E \mapsto \nu(E) = \int_E f \, d\mu
\]

since \( f \) is measurable and non-negative, \( \nu \) is a measure, and we have

\[
\nu(X) = \int_X f \, d\mu < \infty
\]

We assume \( \nu(X) > 0 \), otherwise \( f \) is zero almost everywhere, and the result becomes trivial. Now consider the functions

\[
g_n : X \rightarrow \mathbb{R}_+
\]

\[
x \mapsto g_n(x) = \begin{cases} f_n(x) & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases}
\]

We have

- \( g_n \) is a measurable function
- \( g_n \) converges pointwise to 1 on \( A \) (indeed, if \( x \in A \cap B^c \), then \( g_n(x) = 1 \) if \( x \not\in B \), then \( f_n(x) \to f(x) \to 0 \), thus \( g_n(x) \to 1 = 1 \) if \( x \in A \cap B \cap B^c \), and \( \nu(A^c) = \int_{A^c} f \, d\mu = 0 \) since \( \mu(A^c) = 0 \). Therefore \( g_n \to 1_B \) almost everywhere.

We can now apply Egoroff’s theorem on \( (g_n) \). Let \( \epsilon > 0 \). By Egoroff’s theorem, there exists \( F \in \mathcal{A} \) such that \( \nu(F) \leq \epsilon/4 \) (i.e. \( \int_F f \, d\mu \leq \epsilon/4 \)) and \( (g_n) \) converges uniformly to 1 on \( F^c \). Consider the constant \( c = \frac{\epsilon}{4\nu(X)} \) (note that this is well defined since we assumed \( \nu(X) > 0 \)). Then there exists \( N_1 \) such that for all \( n \geq N_1 \), for all \( x \in F^c \), \( |g_n(x) - f(x)| \leq cf(x) \).

We then have

\[
\int_X |f - f_n| \, d\mu = \int_{B^c} f_n \, d\mu + \int_{B \cap F^c} |f - f_n| \, d\mu + \int_{B \cap F} |f - f_n| \, d\mu
\]

\[
\leq \int_{B^c} f_n \, d\mu + \int_{B \cap F} f \, d\mu + \int_{B \cap F} f_n \, d\mu + \int_{B \cap F^c} |f - f_n| \, d\mu
\]

since \( \int_E f_n \, d\mu \) converges to \( \int_E f \, d\mu \) for all \( E \in \mathcal{A} \) (by 1), there exists \( N_2 \) such that for all \( n \geq N_2 \),

\[
\int_{B^c} f_n \, d\mu \leq \epsilon/4 + \int_{B \cap F} f \, d\mu = \epsilon/4 + 0 \text{ and } \int_{B \cap F} f_n \, d\mu \leq \epsilon/4 + \int_{B \cap F} f \, d\mu \leq \epsilon/4 + \epsilon/4.
\]

Therefore we have for all \( n \geq \max(N_1, N_2) \),

\[
\int_X |f - f_n| \, d\mu \leq 3\epsilon/4 + \int_{B \cap F^c} |f - f_n| \, d\mu
\]

\[
\leq 3\epsilon/4 + c \int_{B \cap F^c} f \, d\mu
\]

\[
\leq 3\epsilon/4 + c \int_X f \, d\mu
\]

\[
= 3\epsilon/4 + \epsilon/4
\]

which proves \( \int_X |f - f_n| \, d\mu \) converges to 0.
\begin{equation}
\lim_{n \to \infty} \int_0^n (1 - \frac{x}{n})^ne^{x/2}dx = 2.
\end{equation}

\textbf{proof} \quad \text{For all } n \in \mathbb{N}, \text{ let }
\begin{align*}
f_n : \mathbb{R}_+ &\to \mathbb{R} \\
x &\mapsto f(x) = (1 - x/n)^ne^{x/2}1_{[0,n]}
\end{align*}
We seek to compute \( \lim_{n \to \infty} \int_{\mathbb{R}_+} f_n(x)dx \). We first observe that for all \( n \in \mathbb{N} \), \( f_n(x) \leq f_{n+1}(x) \).\footnote{\( f_n(x) \leq f_{n+1}(x) \): this is true for \( x > n \). To prove this for \( x \in [0,n] \), consider}
\begin{align*}
h : [0,n) &\to \mathbb{R} \\
x &\mapsto h(x) = \log(f_{n+1}(x)/f_n(x)) \\
&= (n + 1) \log(1 - \frac{x}{n}) - n \log(1 - \frac{x}{n})
\end{align*}
This is well defined since \( f_{n+1}(x)/f_n(x) > 0 \). Then we have for all \( x \in [0,n] \)
\begin{align*}
h'(x) = -\frac{1}{\log(1 - \frac{x}{n+1})} + \frac{1}{\log(1 - \frac{x}{n})} > 0
\end{align*}
therefore \( h \) is increasing, and for all \( x \in [0,n] \), \( h(x) \geq h(0) = 0 \), therefore \( f_{n+1}(x)/f_n(x) \geq 1 \).