MATH 202A - Problem Set 10

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(10.1) Let $\Omega \in \mathbb{R}^n$ be a nonempty open set. Then Ω can be written as the countable union of nested compact sets K_n such that $K_n \subseteq \text{int}(K_{n+1})$

proof Consider any metric

$$\begin{aligned} d : \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R}_+ \\ (x, y) &\mapsto d(x, y) \end{aligned}$$

and let

$$\begin{aligned} \phi : \mathbb{R}^n &\to \mathbb{R} \\ x &\mapsto d(x, E^c) \end{aligned}$$

We first observe that since E^c is closed, $x \in E^c \Leftrightarrow d(x, E^c) = 0$ (indeed, if $x \in E^c$, then $d(x, E^c) = \inf_{y \in E^c} d(x, y) = 0$, and conversely, if $d(x, E^c) = 0$, then there exists a sequence (x_n) of elements of E^c such that $\lim_{n\to\infty} d(x_n, x) = 0$, i.e. (x_n) converges to x. But since E^c is closed, it contains all its limit points, thus $x \in E^c$). Therefore we have

$$x \in E \Leftrightarrow \phi(x) > 0 \tag{1}$$

Let $x_0 \in \Omega$ (this is possible since Ω is nonempty). For all $n \in \mathbb{N} \setminus \{0\}$, let

$$\begin{aligned} E_n &= \{x \in \mathbb{R}^n | \phi(x) \geq 1/n\} \\ B_n &= \{x \in \mathbb{R}^n | d(x, x_0) \leq n\} \\ K_n &= B_n \cap E_n \end{aligned}$$

We have E_n is closed since it is the inverse image by the continuous function ϕ of the closed subset $[1/n, +\infty)$. We also have the closed ball B_n is closed. Therefore K_n is closed as the intersection of two closed subsets. K_n is also bounded since it is a subset of the closed ball B_n. Therefore K_n is compact.

First, we prove that

$$E = \cup_{n\geq1} K_n$$

Indeed, we have

- for all n, $E_n \subseteq E$ since $\forall x \in E_n$, $\phi(x) \geq 1/n > 0$. Therefore for all n, $K_n \subseteq E_n \subseteq E$, which proves that $\cup_{n\geq1} K_n \subseteq E$.
- to prove the second inclusion, let $x \in E$. Then we have $\phi(x) > 0$ by (1). Let $n \in \mathbb{N}$ such that $0 < 1/n \leq \phi(x)$ and $n \geq d(x, x_0)$. Then we have $x \in E_n$ since $\phi(x) \geq 1/n$, and $x \in B_n$ since $d(x, x_0) \leq n$. Therefore $x \in K_n$, and $x \in \cup_{n\geq1} K_n$.

Next, we prove that for all n,

$$K_n \subseteq \text{int}(K_{n+1})$$

Indeed, let $x \in K_n$. Then $\phi(x) \geq 1/n$ and $d(x, x_0) \leq n$. Let $\epsilon > 0$ such that $\epsilon > 1/n - 1/(n+1)$ and $\epsilon < 1$ (this is possible since $1/n - 1/(n+1) < 1$ for $n \geq 1$). Then we have for all $y \in B(x, \epsilon)$

- $d(x_0, y) \leq d(x_0, x) + d(x, y) \leq n + \epsilon < n + 1$, thus $y \in B_{n+1}$

- using the triangle inequality, we have for all $z \in E^c$, $d(y,z) \geq d(x,z) - d(x,y)$, thus taking the infimum, we have $\inf_{z\in E^c} d(y,z) \geq \inf_{z\in E^c} d(x,z) - d(x,y)$ i.e.

$$\phi(y) \geq \phi(x) - d(x,y)$$

Therefore we have

$$\begin{aligned}
\phi(y) &\geq \phi(x) - d(x,y) \\
&\geq 1/n - \epsilon && \text{since } d(x,y) \leq \epsilon \\
&> 1/n - (1/n - 1/(n+1)) \\
&= 1/(n+1)
\end{aligned}$$

therefore $y \in E_{n+1}$.

This proves that $y \in B_{n+1} \cap E_{n+1} = K_{n+1}$, thus

$$B(x,\epsilon) \subseteq K_{n+1}$$

which proves that $x \in \text{int}(K_{n+1})$, and therefore $K_n \subseteq K_{n+1}$, which proves the result.

(10.2) Let X be a topological space. Then the collection $\mathcal{S}$ defined by

$$\mathcal{S} = \{V_{K,g,\epsilon}, K \text{ compact } \subset X,\ g \in C(X), \epsilon > 0\}$$

where

$$V_{K,g,\epsilon} = \{f \in C(X) : \sup_{x\in K} |f(x) - g(x)| < \epsilon\}$$

is a base for the topology on $C(X)$.

proof Let V_{K_i,g_i,ϵ_i}, $i \in \{1,2\}$ be two sets in the collection $\mathcal{S}$, and assume $f \in V_{K_1,g_1,\epsilon_1} \cap V_{K_2,g_2,\epsilon_2}$.

Let $M_i = \sup_{x\in K_i} |f(x) - g_i(x)|$, for $i \in \{1,2\}$, and let $\epsilon = \min(\epsilon_1 - M_1, \epsilon_2 - M_2)$. And let $K = K_1 \cup K_2$. We have

- $\epsilon > 0$ since for $i \in \{1,2\}$, $M_i < \epsilon_i$ ($f \in V_{K_i,g_i,\epsilon_i}$)
- K is compact as the union of two compact sets
- f is continuous by assumption

Therefore $V_{K,f,\epsilon} \in \mathcal{S}$. And we have $V_{K,f,\epsilon} \subseteq V_{K_1,g_1,\epsilon_1} \cap V_{K_2,g_2,\epsilon_2}$. Indeed, let $h \in V_{K,f,\epsilon}$, and let $i \in \{1,2\}$. We have $\forall x \in K_i$, $|h(x) - g_i(x)| \leq |h(x) - f(x)| + |f(x) - g_i(x)|$ by the triangle inequality. Thus taking the supremum over K_i, we have

$$\begin{aligned}
\sup_{x\in K_i} |h(x) - g_i(x)| &\leq \sup_{x\in K_i} \big(|h(x) - f(x)| + |f(x) - g_i(x)|\big) \\
&\leq \sup_{x\in K_i} |h(x) - f(x)| + \sup_{x\in K_i} |f(x) - g_i(x)| \\
&= \sup_{x\in K_i} |h(x) - f(x)| + M_i \\
&\leq \sup_{x\in K} |h(x) - f(x)| + M_i && \text{since } K_i \subseteq K \\
&< \epsilon + M_i && \text{since } h \in V_{K,f,\epsilon} \\
&\leq \epsilon_i - M_i + M_i && \text{by definition of } \epsilon \\
&= \epsilon_i
\end{aligned}$$

Therefore $\sup_{x\in K_i} |h(x) - g_i(x)| < \epsilon_i$, i.e. $h \in V_{K,g_i,\epsilon_i}$, therefore $h \in V_{K_1,g_1,\epsilon_1} \cap V_{K_2,g_2,\epsilon_2}$, which proves that

$$V_{K,f,\epsilon} \subseteq V_{K_1,g_1,\epsilon_1} \cap V_{K_2,g_2,\epsilon_2}$$

(10.3) Let X be a set and let $\mathcal{A}$ be a $\sigma-$algebra of subsets of X. Let $f : X \to \mathbb{R}^*$. Suppose that for all $q \in \mathbb{Q}$, the set $\{x \in X | f(x) \geq q\}$ is measurable (belongs to $\mathcal{A}$). Then f is measurable.

proof For any $q \in \mathbb{Q}$, let X_q denote the set $X_q = \{x \in X | f(x) < q\} = X \setminus \{x \in X | f(x) \geq q\}$. X_q is measurable as the complement of a measurable set.

Let $r \in \mathbb{R}$. Then we have

$$\{x \in X | f(x) \leq r\} = \bigcap_{q \in \mathbb{Q}, q > r} X_q$$

indeed, we have

- if $f(x) \leq r$, then for all $q \in \mathbb{Q}$ such that $q > r$, $f(x) \leq r < q$, thus $x \in \cap_{q \in \mathbb{Q}, q > r} X_q$.
- if $f(x) > r$, then by density of $\mathbb{Q}$, there exists $q_0 \in \mathbb{Q} \cap (r, f(x))$, i.e. there exists $q_0 \in \mathbb{Q}$ such that $q_0 > r$ and $x \notin X_{q_0}$, therefore $x \notin \cap_{q \in \mathbb{Q}, q > r} X_q$.

Therefore $\{x \in X | f(x) \leq r\}$ is the intersection of countably many measurable sets, thus it is measurable. Since this holds for every $r \in \mathbb{R}$, this proves that f is measurable.

(10.4) Let X be a set and let $\mathcal{A}$ be a $\sigma-$algebra of subsets of X. Let $f_n : X \to \mathbb{R}$ be measurable functions. Let $E = \{x \in X | (f_n(x))_n \text{ converges}\}$, and $F = \{x \in X | \lim_{n \to \infty} f_n(x) = +\infty\}$. Then E and F are measurable.

proof E is measurable: for all $n \in \mathbb{N}$, the function

$$\begin{aligned} g_n : X &\to \mathbb{R}^* \\ x &\mapsto g_n(x) = \sup_{k \geq n} f_k(x) \end{aligned}$$

is measurable as the pointwise supremum of countably many measurable functions (using, for example, the identity $\{x \in X | \sup_{k \geq n} f_k(x) \leq c\} = \cap_{k \geq n} \{x \in X | f_k(x) \leq c\}$). Then the function

$$\begin{aligned} g : X &\to \mathbb{R}^* \\ x &\mapsto g(x) = \inf_{n \in \mathbb{N}} g_n(x) = \limsup f_k(x) \end{aligned}$$

is measurable as the pointwise infimum of countably many measurable functions (using, for example, the identity $\{x \in X | \inf_{n \in \mathbb{N}} g_n(x) \geq c\} = \cap_{n \in \mathbb{N}} \{x \in X | g_n(x) \geq c\}$)

Similarly, the functions

$$\begin{aligned} h_n : X &\to \mathbb{R}^* \\ x &\mapsto h_n(x) = \inf_{k \geq n} f_k(x) \end{aligned}$$

and

$$\begin{aligned} h : X &\to \mathbb{R}^* \\ x &\mapsto h(x) = \sup_{n \in \mathbb{N}} h_n(x) = \liminf f_k(x) \end{aligned}$$

are measurable. Next, for a fixed $x \in X$, the sequence $(f_n(x))_n$ converges in $\mathbb{R}$ if and only if $g(x)$ and $h(x)$ are finite and equal[1]

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[1]

- $\Rightarrow$: if $(f_n(x))$ converges to $l \in \mathbb{R}$, then for all $\epsilon > 0$, there exists N such that for all $n \geq N$, $f_n(x) \in [l - \epsilon, l + \epsilon]$, thus for all $n \geq N$, $\sup_{k \geq n} f_k(x) \in [l - \epsilon, l + \epsilon]$ (and similarly for the inf), which proves that $\limsup f_k(x) = \liminf f_k(x) = l$.
- $\Leftarrow$: if $\limsup f_k(x) = \liminf f_k(x) = l \in \mathbb{R}$, then for all $\epsilon > 0$, there exists N_1 such that for all $n \geq N_1$, $\sup_{k \geq n} f_k(x) \leq l + \epsilon$, and there exists N_2 such that for all $n \geq N_2$, $\inf_{k \geq n} f_k(x) \geq l - \epsilon$. Therefore letting $N = \max(N_1, N_2)$, we have for all $n \geq N$,

  $$l - \epsilon \leq \inf_{k \geq N} f_k(x) \leq f_n(x) \leq \sup_{k \geq N} f_k(x) \leq l + \epsilon$$

  thus $|f_n(x) - l| \leq \epsilon$, which proves that $(f_n(x))$ converges to l.

Consider $\tilde{X} = g^{-1}(\mathbb{R}) \cap h^{-1}(\mathbb{R})$, the set of points x where both $\limsup f_n(x)$ and $\liminf f_n(x)$ are finite. $\tilde{X}$ is measurable as the intersection of measurable subsets. Now let $\tilde{g} = g_{|\tilde{X}}$, $\tilde{h} = h_{|\tilde{X}}$. We have both $\tilde{g}$ and $\tilde{h}$ are measurable on the induced algebra $\tilde{\mathcal{A}} = \tilde{X} \cap \mathcal{A} = \{\tilde{X} \cap A, A \in \mathcal{A}\}$[2] [3]. Finally,

$$\begin{aligned} \tilde{f} : \tilde{X} &\to \mathbb{R} \\ x &\mapsto \tilde{f}(x) = \tilde{g}(x) - \tilde{h}(x) \end{aligned}$$

is also measurable since it is well-defined on $\tilde{X}$ (both $\tilde{g}$ and $\tilde{h}$ are finite). Now we can write E as

$$\begin{aligned} E &= \{x \in \tilde{X} | \limsup f_n(x) = \liminf f_n(x)\} \\ &= \{x \in \tilde{X} | \tilde{f}(x) = 0\} \\ &= \tilde{f}^{-1}(\{0\}) \\ &\in \tilde{\mathcal{A}} && \text{since } f^{-1}\{0\} \text{ is measurable with respect to } \tilde{\mathcal{A}} \end{aligned}$$

therefore $E = \tilde{X} \cap A$ for some $A \in \mathcal{A}$. Finally, since $\tilde{X} \in \mathcal{A}$ and $\mathcal{A}$ is closed under intersection, $E \in \mathcal{A}$.

F is measurable: we have $(f_n(x))_n$ converges to $+\infty$ if and only if $h(x) = g(x) = +\infty$, therefore F is simply

$$F = g^{-1}(\{+\infty\}) \cap h^{-1}(\{+\infty\})$$

and is measurable as the intersection of two measurable subsets.[4]

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[2]If $\mathcal{A}$ is a $\sigma-$algebra on X, and $\tilde{X}$ is a subset of X, then $\tilde{\mathcal{A}} = \tilde{X} \cap \mathcal{A}$ is a $\sigma-$algebra on $\tilde{X}$. Indeed:

- $\emptyset \in \tilde{\mathcal{A}}$ since $\tilde{X} \cap \emptyset = \emptyset$.
- if $\tilde{A} \in \tilde{\mathcal{A}}$, then there exists $A \in \mathcal{A}$ such that $\tilde{A} = \tilde{X} \cap A$, and we have $\tilde{X} \setminus \tilde{A} = \tilde{X} \cap (X \setminus A)$, where $X \setminus A \in \mathcal{A}$, thus $\tilde{X} \setminus \tilde{A} \in \tilde{\mathcal{A}}$.
- if $(\tilde{A_n})$ is a sequence of elements in $\tilde{\mathcal{A}}$, then there exist $A_n \in \mathcal{A}$ such that $\tilde{A_n} = \tilde{X} \cap A_n$, and we have $\cap_n \tilde{A_n} = \tilde{X} \cap (\cap_n A_n)$, where $\cap_n A_n \in \mathcal{A}$, thus $\cap_n \tilde{A_n} \in \tilde{\mathcal{A}}$.

[3]if $f : X \to \mathbb{R}^*$ is measurable and $\tilde{X} \subseteq X$, then the restrictions $\tilde{f} = f_{|\tilde{X}}$ is measurable on the induced algebra $\tilde{\mathcal{A}} = \tilde{X} \cap \mathcal{A}$: we have

$$\tilde{f}^{-1}([-\infty, c]) = \{x \in \tilde{X} : f(x) \leq c\} = \tilde{X} \cap f^{-1}([-\infty, c]) \in \tilde{X} \cap \mathcal{A}$$

[4]using

$$g^{-1}\{+\infty\} = \cap_{n \in \mathbb{N}} g^{-1}([n, +\infty])$$

(10.5) Let X be a set and consider a sequence (E_n) of subsets of X. Let $A = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k$. Then we have $\forall x \in X$,

$$1_A(x) = \limsup 1_{E_n}(x)$$

proof Let $x \in X$. First, we observe that since $(\sup_{k \geq n} 1_{E_k}(x))_n$ is a non-increasing sequence,

$$\lim_{n\to\infty} \sup_{k\geq n} 1_{E_k}(x) = \inf_{n\in\mathbb{N}} \sup_{k\geq n} 1_{E_k}(x)$$

We also observe that since for all $k \in \mathbb{N}$, $1_{E_k}(x) \in [0, 1]$, we have (taking the supremum over $k \geq n$ then the infimum over $n \in \mathbb{N}$)

$$\limsup 1_{E_k}(x) \in [0, 1] \tag{2}$$

We have for all $x \in X$

$$\begin{aligned}
1_A(x) = 1 &\Rightarrow x \in \cap_{n=1}^{\infty} \cup_{k\geq n} E_k \\
&\Rightarrow \forall n \in \mathbb{N}, \exists k \geq n : x \in E_k \\
&\Rightarrow \forall n \in \mathbb{N}, \exists k \geq n : 1_{E_k}(x) = 1 \\
&\Rightarrow \forall n \in \mathbb{N}, \sup_{k\geq n} 1_{E_k}(x) \geq 1 \\
&\Rightarrow \inf_{n\in\mathbb{N}} \sup_{k\geq n} 1_{E_k}(x) \geq 1 \\
&\Rightarrow \limsup 1_{E_n}(x) \geq 1 \\
&\Rightarrow \limsup 1_{E_n}(x) = 1 && \text{since } \limsup 1_{E_k}(x) \leq 1 \text{ by (2)}
\end{aligned}$$

and we have

$$\begin{aligned}
1_A(x) = 0 &\Rightarrow x \notin \cap_{n=1}^{\infty} \cup_{k\geq n} E_k \\
&\Rightarrow \exists n \in \mathbb{N}, \forall k \geq n : x \notin E_k \\
&\Rightarrow \exists n \in \mathbb{N}, \forall k \geq n : 1_{E_k}(x) = 0 \\
&\Rightarrow \exists n \in \mathbb{N}, \sup_{k\geq n} 1_{E_k}(x) = 0 \\
&\Rightarrow \inf_{n\in\mathbb{N}} \sup_{k\geq n} 1_{E_k}(x) \leq 0 \\
&\Rightarrow \limsup 1_{E_k}(x) \leq 0 \\
&\Rightarrow \limsup 1_{E_k}(x) = 0 && \text{since } \limsup 1_{E_k}(x) \geq 0 \text{ by (2)}
\end{aligned}$$