MATH 202A - Problem Set 1

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August 29, 2012

(1.1) Let X be an infinite set.

Let F be the set of injective functions from a segment to X, where a segment is a subset of $\mathbb{N}$ that is either of the form $\{1, \dots, n_0\}$ for some $n_0 \in \mathbb{N}$, or the entire set $\mathbb{N}$.

- Any element $f \in F$ is a subset of $\mathbb{N} \times X$ (so $F \subseteq \mathcal{P}(\mathbb{N} \times X)$), thus F is partially ordered for upper inclusion $\subseteq$.
- F is nonempty since it contains $\{1\} \to X, 1 \mapsto x$ where x is any element in X
- For any chain $C \subseteq F$, the union $\cup_{f \in C} f$ is an upper bound (since every element of the chain is a subset of the union).

Therefore by Zorn's Lemma, F has a maximal element. Let $f_0 : N_0 \to X$ be maximal in F, where N_0 is a segment. Then we must have $N_0 = \mathbb{N}$: assume that is not the case, by contradiction. Then $\exists n_0 \in \mathbb{N}$ such that $N_0 = \{1, \dots, n_0\}$, and the image of N_0 by f_0, $f_0(N_0) = \{f_0(n), n \in N_0\}$, is finite, thus $X - f_0(N_0)$ is nonempty (since X is infinite). Let $x_0 \in X - f_0(N_0)$. Then $f_1 = f_0 \cup (n_0 + 1, x_0)$ is injective by construction, and is an element of F that strictly contains f_0, which contradicts maximality of f_0. Therefore $N_0 = \mathbb{N}$.

Therefore $f_0 : \mathbb{N} \to X$ is injective, and

$$\begin{aligned} \tilde{f}_0 : \mathbb{N} &\to f_0(\mathbb{N}) \\ n &\mapsto f_0(n) \end{aligned}$$

is bijective, thus $f_0(\mathbb{N}) \subseteq X$ is countably infinite.

(1.2) Let S be a set and $\mathcal{P}(S)$ be its power set. Let $f : S \to \mathcal{P}(S)$ be any function from S on $\mathcal{P}(S)$. Let $X \subseteq S$ be the set defined by $X = \{s \in S | s \notin f(s)\}$. If f is surjective, then $\exists x \in S$ such that $f(x) = X$. This leads to a contradiction since

- if $x \in X$, then we must have $x \notin f(x)$ (by definition of X), i.e. $x \notin X$.
- if $x \notin X$, then we must have $x \in f(x)$ (by definition of X), i.e. $x \in X$.

Therefore there is no surjective function from S onto $\mathcal{P}(S)$.

(1.3) Let A be a countable set, and suppose that for each $a \in A$, S_a is a countable set. Let $S = \cup_{a\in A} S_a$. An equivalent definition[1] of countable set is: S countable if and only if there exists a surjection from $\mathbb{N}$ onto S.

$\forall a \in A$, let F_a be the set of surjective functions from $\mathbb{N}$ to S_a (F_a is nonempty since S_a is countable). By the axiom of choice, let $(f_a)_{a\in A}$ be a sequence of such surjective functions (element of $\times_{a\in A} F_a$).

Now define the function f as

$$\begin{aligned} f : A \times \mathbb{N} &\to S \\ (a, n) &\mapsto f_a(n) \end{aligned}$$

f is surjective since $\forall s \in S$, $\exists a \in A$ such that $s \in S_a$, and since $f_a : \mathbb{N} \to S_a$ is surjective, $\exists n \in \mathbb{N}$ such that $f_a(n) = s$, and we have $f(a, n) = f_a(n) = s$.

Since A is countable, there exists a surjective function $g : \mathbb{N} \to A$, and the function $\tilde{g}$ defined as

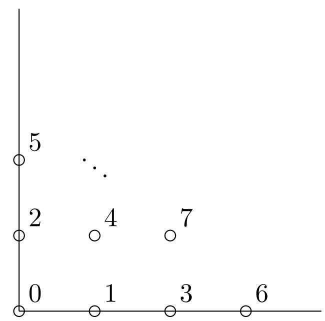
$$\begin{aligned} \tilde{g} : \mathbb{N} \times \mathbb{N} &\to A \times \mathbb{N} \\ (n, m) &\mapsto (g(n), m) \end{aligned}$$

is surjective ($\forall (a, m) \in A \times \mathbb{N}$, since g is surjective $\exists n \in \mathbb{N}$ such that $g(n) = a$, and $\tilde{g}(n, m) = (g(n), m) = (a, m)$).

Finally, since $\mathbb{N} \times \mathbb{N}$ is countable, there exists a surjective function $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. For example, such a function is given by

$$\begin{aligned} h(0) &= (0, 0) \\ h(1) &= (1, 0) \\ h(2) &= (0, 1) \\ h(3) &= (2, 0) \\ h(4) &= (1, 1) \\ h(5) &= (0, 2) \end{aligned}$$

and so on.



The function $f \circ \tilde{g} \circ h : \mathbb{N} \to S$ is surjective as composition of surjective functions, therefore S is countable.

$$\mathbb{N} \xrightarrow{h} \mathbb{N} \times \mathbb{N} \xrightarrow{\tilde{g}} A \times \mathbb{N} \xrightarrow{f} S$$

(1.4) Let X be a non-empty set and $f, g : X \to \mathbb{R}$ be bounded functions. Let $a = \sup_{x\in X} f(x)$ and $b = \sup_{x\in X} g(x)$. We have by definition of the supremum, $\forall x \in X$ $f(x) \leq a$ and $g(x) \leq b$ (the supremum is

[1] proof: $\Rightarrow$ Let S be a countable set. If S is finite, let $n_0 = |S|$, and let $S = \{s_0, \ldots, s_{n_0-1}\}$. The function

$$\begin{aligned} j : \mathbb{N} &\to S \\ n &\mapsto s_{\min(n, n_0-1)} \end{aligned}$$

is clearly surjective. If S is infinite, then there exists an bijection $b : \mathbb{N} \to S$, which is, in particular, surjective.

$\Leftarrow$ Let S be a set, and assume that there exists a surjection $j : \mathbb{N} \to S$. If S is finite, then S is countable. If S is infinite, then

- there exists an injection from $\mathbb{N}$ to S (by (1.1))
- we can construct an injection from S to $\mathbb{N}$ as follows:

$$\begin{aligned} i : S &\to \mathbb{N} \\ s &\mapsto \min N_s \end{aligned}$$

where $N_s = \{n \in N | j(n) = s\}$ is nonempty for all $s \in S$ (since j is surjective) and $N_s \cap N_{s'} = \emptyset$ for $s \neq s'$ since j is a function. This proves that $\min N_s \neq \min N_{s'}$ for $s \neq s'$, i.e. i is injective.

Therefore by the Cantor-Bernstein theorem, there exists a bijection $b : \mathbb{N} \to S$, and S is countable an upper bound). Thus

$$\forall x \in X,\ f(x) + g(x) \le a + b$$

and $a + b$ is an upper bound of the function

$$\begin{aligned} f + g : X &\to \mathbb{R} \\ x &\mapsto f(x) + g(x) \end{aligned}$$

thus $\sup_{x\in X}(f(x) + g(x))$ exists, and satisfies $\sup_{x\in X}(f(x)+g(x)) \le a+b$ (the supremum is the least upper bound).

(1.5) Let (a_n) be a monotone increasing bounded sequence in $\mathbb{R}$. Since (a_n) is bounded, $\sup_{n\in\mathbb{N}} a_n$ exists. Let $a = \sup_{n\in\mathbb{N}} a_n$. We show that (a_n) converges and that its limit is a.

We have $\forall \epsilon > 0$, $\exists N_\epsilon$ such that $a_{N_\epsilon} > a - \epsilon$ (otherwise $a - \epsilon$ is an upper bound of (a_n) which contradicts minimality of a). And we have by monotonicity of (a_n), $\forall n \ge N_\epsilon$ $a_n \ge a_{N_\epsilon}$, and by definition of a, $a_n \le a$, thus $a - \epsilon < a_n \le a$, thus $|a - a_n| \le \epsilon$.

Therefore we have

$$\forall \epsilon > 0, \exists N_\epsilon \text{ such that } \forall n \ge N_\epsilon,\ |a_n - a| \le \epsilon$$

i.e. (a_n) converges and its limit is a.

(1.6) Let (a_n) and (b_n) be sequences in $\mathbb{R}$ and suppose $\limsup_{n\to\infty} a_n > -\infty$ and $\limsup_{n\to\infty} b_n > -\infty$. Since $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\ge n} a_k$, and $(\sup_{k\ge n} a_k)_n$ is monotone decreasing, $(\sup_{k\ge n} a_k)_n$ must be bounded. Similarly, $(\sup_{k\ge n} b_k)_n$

First, we have that

$$\forall n \in \mathbb{N},\ \sup_{k\ge n}(a_k + b_k) \le \sup_{k\ge n} a_k + \sup_{k\ge n} b_k \tag{1}$$

(for example by applying (1.4) to the functions $f_n : X_n \to \mathbb{R}, k \mapsto f_n(k) = a_k$ and $f_n : X_n \to \mathbb{R}, k \mapsto g_n(x) = b_k$ where $X_n = \{k \in \mathbb{N} | k \ge n\}$)

The limit $\lim_{n\to\infty} \sup_{k\ge n}(a_k + b_k)$ exists in $\mathbb{R}^*$ since $(\sup_{k\ge n}(a_k + b_k))_n$ is monotone decreasing.

The limit $\lim_{n\to\infty}(\sup_{k\ge n} a_k + \sup_{k\ge n} b_k)$ exists in $\mathbb{R}$ since $(\sup_{k\ge n} a_k + \sup_{k\ge n} b_k)_n$ is monotone decreasing and bounded (as the sum of two monotone decreasing bounded sequences) and we have $\lim_{n\to\infty}(\sup_{k\ge n} a_k + \sup_{k\ge n} b_k) = \lim_{n\to\infty} \sup_{k\ge n} a_k + \lim_{n\to\infty} \sup_{k\ge n} b_k$.

Finally, taking the limit as $n \to \infty$ in (1), we have

$$\lim_{n\to\infty} \sup_{k\ge n}(a_k + b_k) \le \lim_{n\to\infty} (\sup_{k\ge n} a_k + \sup_{k\ge n} b_k) = \lim_{n\to\infty} \sup_{k\ge n} a_k + \lim_{n\to\infty} \sup_{k\ge n} b_k$$

which proves the desired result.

example of strict inequality Consider the following sequences (a_n) and (b_n)

$$a_n = \begin{cases} 0 \text{ if } n \text{ is even} \\ -n \text{ if } n \text{ is odd} \end{cases}$$

$$b_n = \begin{cases} -n \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases}$$

We have $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 0$, but $a_n + b_n = -n\ \forall n$, thus $\limsup_{n\to\infty} a_n + b_n = -\infty$.