MATH 202B - Problem Set 7

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(7.1) Let X be a compact Hausdorff space and let μ be a non negative Radon measure on X. Show that for any Borel measurable function $f : X \to \mathbb{C}$, and any $\epsilon > 0$, there exists a compact set $K \subset X$ such that $f_{|K}$ is continuous, and $\mu(X \setminus K) < \epsilon$.

proof Since X is compact and μ is a Radon measure, then $\mu(X) < \infty$.

First consider the case where f is a simple function

$$f = \sum_{n=1}^{N} c_n 1_{E_n}$$

where E_n are measurable sets. Let $\epsilon > 0$. Since μ is outer regular, and inner regular on measurable sets of finite measure, there exist an open set O_n and a compact set K_n such that

$$\begin{aligned} &K_n \subset E_n \subset O_n \\ &\mu(E_n) - \frac{1}{2N}\epsilon < \mu(K_n) \leq \mu(E_n) \leq \mu(O_n) < \mu(E_n) + \frac{1}{2N}\epsilon \end{aligned}$$

thus $K_n \subset O_n$ and $\mu(O_n) - \mu(K_n) < \frac{1}{N}\epsilon$, and since X is of finite measure, we have

$$\mu(O_n \setminus K_n) < \frac{1}{N}\epsilon$$

consider the closed sets K_n and O_n^c. By Urysohn's Lemma, there exists a continuous function $g_n : X \to [0,1]$ such that $g_{n|K_n} \equiv 1$, and $g_{n|O_n^c} \equiv 0$. Therefore we have for all $x \in K_n \cup O_n^c$, $g_n(x) = 1_{E_n}(x)$.

Since X is compact and O_n^c is closed, it is also compact, and so is $K_n \cup O_n^c$. Now let $K = \cap_{n=1}^{N}(K_n \cup O_n^c)$. K is compact as the finite intersection of compact sets. Consider the function $g = \sum_{n=1}^{N} c_n g_n$. It is continuous as the finite sum of continuous functions, and by definition of f_n, we have for all $x \in K$,

$$g(x) = \sum_{n=1}^{N} c_n g_n(x) = \sum_{n=1}^{N} c_n 1_{E_n}(x) = f(x)$$

therefore $f_{|K}$ is continuous (since it is equal to $g_{|K}$), and

$$\mu(K^c) = \mu(\cup_{n=1}^{N} O_n \setminus K_n) \leq \sum_{n=1}^{N} \mu(O_n \setminus K_n) \leq N\frac{1}{N}\epsilon$$

Now consider the case of a general Borel measurable function f, and let (s_n) be a sequence of simple functions that converge pointwise to f. For all $n \in \mathbb{N}$, by the first part, there exists a compact set K_n such that $\mu(K_n^c) \leq \frac{1}{2^n}\epsilon$, and $s_{n|K_n}$ is continuous.

Since (s_n) converges pointwise to f, by Egoroff's theorem, there exists a measurable $E \subset X$ such that $\mu(E^c) \leq \epsilon$ and $(s_{n|E})$ converges uniformly to $f_{|E}$. By inner regularity of μ on measurable sets of finite measure, there exists a compact $K_E \subset E$ such that $\mu(K_E) \geq \mu(E) - \epsilon$, thus $\mu(E \setminus K_E) \leq \epsilon$, therefore

$$\mu(K_E^c) = \mu(E^c \cup (E \setminus K_E)) \leq \mu(E^c) + \mu(E \setminus K_E) \leq 2\epsilon$$

finally, consider the compact set $K = K_E \cap (\cap_{n \in \mathbb{N}} K_n)$. We have

$$\begin{aligned} \mu(K^c) &\leq \mu(K_E^c) + \sum_{n \in \mathbb{N}} \mu(K_n^c) \\ &\leq 2\epsilon + \epsilon \sum_{n \in \mathbb{N}} \frac{1}{2^n} \\ &= 4\epsilon \end{aligned}$$

and $(s_{n|K})$ is a sequence of continuous functions (since $K \subset K_n$ for all n) that converges uniformly to $f_{|K}$ (since $K \subset K_E$). Therefore $f_{|K}$ is continuous as the uniform limit of continuous functions, and K^c has measure $\leq 4\epsilon$.

(7.2) This problem establishes the rotation invariance of the Borel measure $\mu = d\omega$ on the Sphere $S^2 \subset \mathbb{R}^3$ obtained from Riemann integration with respect to $\sin\theta_1 d\theta_1 d\theta_2$ where θ_1 and θ_2 are latitude and longitude with $0 \leq \theta_1 \leq \pi$ and $0 \leq \theta_2 \leq 2\pi$. The measure $d\omega$ is constructed by means of the Riesz representation Theorem, and satisfies for all $f \in C(S^2)$

$$\int_{\theta_1 \in [0,\pi]} \int_{\theta_2 \in [0,2\pi]} f(\Omega(\theta_1, \theta_2)) \sin\theta_1 d\theta_1 d\theta_2 = \int_{S^2} f(\omega) d\mu(\omega)$$

here

$$\begin{aligned} \Omega : [0,\pi] \times [0, 2\pi] &\to S^2 \\ (\theta_1, \theta_2) &\mapsto \Omega(\theta_1, \theta_2) = \begin{pmatrix} \cos(\theta_1)\cos(\theta_2) \\ \cos(\theta_1)\sin(\theta_2) \\ \sin(\theta_1) \end{pmatrix} \end{aligned}$$

(the construction in the book used an abuse of notation which I found extremely confusing, by using both $f(\omega)$ and $f(\theta_1, \theta_2)$. These functions are not the same, one is obtained by a change of variable $f \circ \Omega$)

1. A rotation in $\mathbb{R}^3$ is the linear function L determined by a matrix A with $AA^t = 1$ and $\det A = 1$. For $0 < a < 1 < b < \infty$, let $S_{a,b} = \{x \in \mathbb{R}^3 : r \in (a,b), \theta_1 \in [0,\pi], \theta_2 \in [0, 2\pi]\}$. Show that $L(S_{a,b}) = S_{a,b}$.

   proof For all $x \in S_{a,b}$, $\|x\|_2 \in (a,b)$, and we have

   $$\|Lx\|_2^2 = (Lx)^t Lx = x^t L^t Lx = x^t x = \|x\|_2^2$$

   (in other words the rotation preserves the 2-norm). Therefore $\|Lx\|_2 \in (a,b)$, i.e. $Lx \in S_{a,b}$, therefore $L(S_{a,b}) \subset S_{a,b}$.

Conversely, for all $y \in S_{a,b}$, we have $L^t y \in S_{a,b}$ (since L^t is also a rotation), therefore $y = Lx$ where $x = L^t y \in S_{a,b}$. Thus $S_{a,b} \subset L(S_{a,b})$, therefore we have equality.

2. For any bounded Borel function $F : S_{a,b} \to \mathbb{C}$, and rotation L, let

$$\begin{aligned} LF : S_{a,b} &\to \mathbb{C} \\ x &\mapsto (LF)(x) = F(L^{-1}x) \end{aligned}$$

Prove that

$$\int_{S_{a,b}} (LF) dx = \int_{S_{a,b}} F dx$$

proof Consider the homeomorphism

$$\begin{aligned} L : S_{a,b} &\to S_{a,b} \\ x &\mapsto Lx \end{aligned}$$

since L is a rotation, we have $|\det L| = 1$. And since $L(S_{a,b}) = S_{a,b}$, we have by the change of variable theorem

$$\begin{aligned} \int_{S_{a,b}} (LF) dx &= \int_{S_{a,b}} F \circ L^{-1} dx \\ &= \int_{S_{a,b}} F |\det L| dx && \text{by the change of variable theorem} \\ &= \int_{S_{a,b}} F dx \end{aligned}$$

3. Let $f : S^2 \to \mathbb{C}$ be any continuous function, and define $(LF)(\omega) = F(L^{-1}\omega)$. Extend f to a function

$$\begin{aligned} F : S_{a,b} &\to \mathbb{C} \\ r\omega &\mapsto F(r\omega) = f(\omega) \end{aligned}$$

Prove that

$$\int_{S_{a,b}} F dx = \int_{a,b} r^2 dr \int_{S^2} f(\omega) d\omega$$

Deduce that

$$\int_{S^2} Lf d\omega = \int_{S^2} f d\omega$$

proof Consider the homeomorphism $\Omega : [0, \pi] \times [0, 2\pi] \to S_2$ defined above, and let

$$\begin{aligned} \Omega_{a,b} : (a, b) \times (0, \pi) \times (0, 2\pi) &\to S_{a,b} \\ (r, \theta_1, \theta_2) &\mapsto \Omega_{a,b}(r, \theta_1, \theta_2) = r\Omega(\theta_1, \theta_2) \end{aligned}$$

We have $|\det \Omega_{a,b}(r, \theta_1, \theta_2)| = r^2 \sin\theta_1$. Therefore by the change of variable theorem, we have[1]

$$\begin{aligned} \int_{S_{a,b}} F dx &= \int_{(a,b)} \int_{(0,\pi)} \int_{(0,2\pi)} F(r\Omega(\theta_1, \theta_2)) r^2 \sin\theta_1 d\theta_1 d\theta_2 dr \\ &= \int_{(a,b)} \int_{(0,\pi)} \int_{(0,2\pi)} f(\Omega(\theta_1, \theta_2)) r^2 \sin\theta_1 d\theta_1 d\theta_2 dr && \text{by definition of } F \\ &= \int_{(a,b)} r^2 dr \int_{(0,\pi)} \int_{(0,2\pi)} f(\Omega(\theta_1, \theta_2)) \sin\theta_1 d\theta_1 d\theta_2 \\ &= \int_{(a,b)} r^2 dr \int_{S^2} f(\omega) d\mu(\omega) && \text{by definition of } \mu \text{ (i.e. } d\omega \text{ as denoted in the book)} \end{aligned}$$

[1] $\Omega_{a,b}((a, b) \times (0, \pi) \times (0, 2\pi))$ is $S_{a,b}$ minus the equator, which is a null set

Similarly, we have

$$\int_{S_{a,b}} (LF)dx = \int_{(a,b)} r^2 dr \int_{S^2} (Lf)(\omega)d\mu(\omega)$$

obtained by applying the previous result to the functions $LF : S_{a,b} \to \mathbb{C}$ and $Lf : S^2 \to \mathbb{C}$, which satisfy

$$\begin{aligned}
(LF)(rw) &= F(L^{-1}(rw)) \\
&= F(rL^{-1}(w)) && \text{by linearity of } L \\
&= f(L^{-1}(w)) \\
&= (Lf)(w)
\end{aligned}$$

Finally, combining the previous results,

$$\begin{aligned}
\int_{S^2} Lf d\omega &= \frac{\int_{S_{a,b}} F dx}{\int_{(a,b)} r^2 dr} \\
&= \frac{\int_{S_{a,b}} (LF) dx}{\int_{(a,b)} r^2 dr} && \text{by (b)} \\
&= \int_{S^2} f d\omega
\end{aligned}$$

4. Deduce that for every Borel $E \subset S^2$

$$d\omega(L(E)) = d\omega(E)$$

proof define the Radon measure

$$\nu(E) = \mu(L(E))$$

(or $\nu(E) = d\omega(E)$), and consider the positive linear functionals defined on $C(S_{a,b})$

$$\ell_\nu(f) = \int_{S^2} f d\nu$$

$$\ell_\mu(f) = \int_{S^2} f d\mu$$

Claim: $\forall f \in C(S^2)$, $\ell_\nu(f) = \ell_\mu(Lf) = \ell_\mu(f)$. As a consequence of the claim, by the uniqueness of the Riesz Representation, we have the two Radon measures are equal, i.e. $\nu = \mu$.

As a preliminary, we first show that for indicator functions of measurable subsets $E \subset S^2$, $\ell_\nu(1_E) = \ell_\mu(L1_E)$. We have

$$\begin{aligned}
\ell_\nu(1_E) &= \nu(E) \\
&= \mu(L(E)) \\
&= \int_{S^2} 1_{L(E)}(\omega)d\mu(\omega) \\
&= \int_{S^2} 1_E(L^{-1}(\omega))d\mu(\omega) \\
&= \int_{S^2} (L1_E)(\omega)d\mu(\omega) \\
&= \ell_\mu(L1_E)
\end{aligned}$$

the result follows for simple functions by linearity of ℓ_ν, ℓ_μ, and the operator $f \mapsto Lf$. To show the claim, let $f \in C(S^2)$. Since f is continuous, it is measurable, let (s_n) be a sequence of simple functions that converge pointwise to f, $s_n : S^2 \to \mathbb{C}$, and such that $|s_n| \le |f|$. Thus we can apply the Dominated convergence theorem to obtain

$$\ell_\nu(f) = \lim_n \ell_\nu(s_n)$$

since we have the domination $|\int_{S^2} s_n d\nu| \le \int_{S^2} |f| d\nu$, and f is bounded (continuous on the compact S^2). We also have (Ls_n) is a sequence of simple functions that converges to Lf, and $|Ls_n|$ is dominated by $|Lf|$, and we can apply the DCT. Finally,

$$\begin{aligned}
\ell_\nu(f) &= \lim_n \ell_\nu(s_n) && \text{by the DCT}\\
&= \lim_n \ell_\mu(Ls_n) && \text{by the simple function case}\\
&= \ell_\mu(Lf) && \text{by the DCT}\\
&= \ell_\mu(f) && \text{by (c)}
\end{aligned}$$

(7.3) Let X be a locally compact Hausdorff space. Let μ be a positive Radon measure on X. Let $h \in L^1(X)$. Show that $\nu(E) = \int_E h d\mu$ defines a signed Radon measure ν on X.

proof First, we show that $\nu^+(E) = \sup_{F \subset E} \nu(F) = \int_E h^+ d\mu$. Indeed, we have for all measurable $F \subset E$,

$$\nu(F) = \int_F h d\mu \le \int_F h^+ d\mu \le \int_E h^+ d\mu$$

thus $\nu(E) \le \int_E h^+ d\mu$. Conversely, taking $F = h^{-1}([0, \infty])$, we have F is a measurable subset of E, thus

$$\nu(E) \ge \nu(F) = \int_F h d\mu = \int_E h^+ d\mu$$

which proves equality. Similarly, we have $\nu^-(E) = \int_E h^- d\mu$, and it suffices to prove that ν^+ and ν^- are Radon measures.

First, we observe that since $h^+ \in L^1(X)$, the set $A = \{x \in X : h^+(x) > 0$ is $\sigma-$finite, since

$$A = \cup_{n \in \mathbb{N}} A_n$$

where $A_n = \{x \in X : h^+(x) > 1/n\}$ and $\mu(A_n) \le n\|h^+\|_1$ by the Chebyshev inequality. Therefore we have by the Monotone Convergence Theorem

$$\begin{aligned}
\nu^+(E) &= \int_E \lim_n h^+ 1_{A_n} d\mu\\
&= \lim_n \int_E h^+ 1_{A_n} d\mu && \text{by the MCT, since the } A_n \text{ are nested}\\
&= \lim_n \int_{E \cap A_n} h^+ d\mu
\end{aligned}$$

- inner regularity of μ^+ on open sets: let E be an open set. Let $\epsilon > 0$. We showed that $\nu^+(E) = \lim_n \int_{E\cap A_n} h^+ d\mu$, thus there exists N such that $0 \leq \nu^+(E) - \int_{E\cap A_N} h^+ d\mu \leq \epsilon/2$.

  By inner regularity of μ on $E \cap A_N$ (measurable set of finite measure), there exists compact $K_N \subset (E \cap A_N)$ such that $\mu(K_N) \geq \mu(E \cap A_N) - \delta_0$, thus since $\mu(E \cap A_N)$ is finite,

  $$\mu((E \cap A_N) \setminus K_N) \leq \delta_0$$

  thus

  $$\int_{E\cap A_N} h^+ d\mu - \int_{K_N} h^+ d\mu \leq \epsilon/2$$

  and we have $K_N \subset (E \cap A)$, thus

  $$\begin{aligned} \geq \int_{K_N} h^+ d\mu &\geq \int_{E\cap A_N} h^+ d\mu - \epsilon/2 \\ &\geq \nu^+(E) - \epsilon/2 - \epsilon/2 \end{aligned}$$

  i.e.

  $$\nu^+(E) \geq \nu^+(K_N) \geq \nu^+(E) - \epsilon$$

  which proves inner regularity (on open sets)

- outer regularity of μ^+: let E be a measurable set. Let $\epsilon > 0$.

  Since $h^+ \in L^1(X)$, for all n, there exists $\delta_n > 0$ such that for all measurable F, $\mu(F) \leq \delta_n \Rightarrow \int_F h^+ d\mu \leq \frac{1}{2^{n+1}}\epsilon$. By outer regularity of μ, there exists $O_n \supset (E \cap (A_n \setminus A_{n-1}))$ such that $\mu(O_n) \leq \mu(E \cap (A_n \setminus A_{n-1})) + \delta_n$. Thus (since $E \cap (A_n \setminus A_{n-1})$ has finite μ-measure)

  $$\mu(O_n \setminus (E \cap (A_n \setminus A_{n-1}))) \leq \delta_n$$

  thus

  $$\int_{O_n} h^+ d\mu - \int_{E\cap(A_n\setminus A_{n-1})} h^+ d\mu \leq \frac{1}{2^{n+1}}\epsilon$$

  now consider the open set $O = \cup_{n\in\mathbb{N}} O_n$. We have $O \supset (E \cap A)$, thus

  $$\begin{aligned} \int_{E\cap A} h^+ d\mu &\leq \int_O h^+ d\mu \\ &= \lim_n \int_{\cup_{k\leq n} O_k} h^+ d\mu && \text{by the MCT} \\ &\leq \lim_n \int_{E\cap A_n} h^+ d\mu + \epsilon \sum_{n\in\mathbb{N}} \frac{1}{2^{n+1}} \\ &= \int_{E\cap A} h^+ d\mu + \epsilon \end{aligned}$$

  i.e.

  $$\nu^+(E) \leq \nu^+(O) \leq \nu^+(E) + \epsilon$$

  which proves outer regularity

(the same arguments apply to μ^-)

(7.4) Here is an example of a Radon measure which is not inner regular on arbitrary sets. Let X be the locally compact Hausdorff space $X = \mathbb{R} \times \mathbb{R}'$ (the product of two topological spaces) where the second factor $\mathbb{R}'$ is $\mathbb{R}$ equipped with the discrete topology. Write $f^y(x) = f(x, y)$ and $E^y = \{x : (x, y) \in E\}$

1. Show that $f \in C_c(X)$ if and only if $f^y \in C_c(\mathbb{R})$ for all $y \in \mathbb{R}'$, and $Y(f) = \{y : f^y \not\equiv 0\}$ is finite.

   proof First, we show that

   $$K \text{ is a compact subset of } X \Leftrightarrow \forall y \in \mathbb{R}',\ K^y \text{ is compact, and } Y(K) = \{y : K^y \neq \emptyset\} \text{ is finite}$$

   proof

   - suppose K is compact. Let $y \in \mathbb{R}'$ and let (U_n) be an open cover of K^y. Then $(U_n \times \mathbb{R}')$ is an open cover for K, and since K is compact, there exists a finite subcover $(U_{n_i} \times \mathbb{R}')$, and (U_{n_i}) is thus a finite subcover of K^y, which shows that K^y is compact. To show that $Y(K)$ is finite, assume by contradiction that it is not. Then consider the open cover of K given by $(\mathbb{R} \times \{y\})_{y \in Y(K)}$. No finite subcover exists, since for all $y \in Y(K)$, K^y is nonempty, and is only covered by $\mathbb{R} \times \{y\}$ (thus the cover has no redundancy), contradiction.
   - suppose K^y is compact for all y, and $Y(K)$ is finite. Let (U_n) be an open cover of K, then for all $y_i \in Y(K)$ $(i \in \{1, \ldots, I\})$, $U_n^{y_i}$ is an open cover, and since K^y is compact, there exists a finite cover $U_{n_i}^{y_i}$. Then
   $$K \subset \cup_{i=1}^{I} \cup_{n_i} K_{n_i}$$
   and we have a finite cover of K, which shows that K is compact

   Now we show the result for f

   - Let $f \in C_c(X)$. Let $K \subset X$ be a compact set such that f is zero on K^c.
   By the above, we have K^y is compact for all $y \in \mathbb{R}'$, and $Y(K)$ is finite.
   Then for all $y \in \mathbb{R}'$, f^y is continuous (if O is an open subset of $\mathbb{R}$, then $(f^y)^{-1}(O) = (f^{-1}(O))^y$ is open) and f^y is 0 on $(K^y)^c$. Since K^y is compact, $f^y \in C_c(\mathbb{R})$. And we have for all $y \notin Y(K)$, K^y is empty, thus $f^y \equiv 0$. Therefore $f^y \not\equiv 0$ for finitely many y.
   - Let f such that $f^y \in C_c(\mathbb{R})$ for all y, and $Y(f) = \{y : f^y \not\equiv 0\}$ is finite. For all y let K^y compact such that f^y is zero on $(K^y)^c$. Then $K = \cup_{y \in Y(f)} K^y \times \{y\}$ is compact (by the previous discussion) and $f \in C_c(K)$ (continuity follows from $f^{-1}(O) = \cup_y (f^y)^{-1}(O) \times \{y\}$).

2. Define the positive linear functional

   $$\ell : C_c(X) \to \mathbb{R}_+$$
   $$f \mapsto \ell(f) = \sum_{y \in \mathbb{R}'} \int_{\mathbb{R}} f^y(x) dm(x)$$

   Let μ be the Radon measure on X associated to ℓ via the first Riesz Representation theorem. Show that for every Borel set $E \subset X$, $\mu(E) = \infty$ unless $Y(E) = \{y \in \mathbb{R}' : E^y \neq \emptyset\}$ is countable.

   proof Let E be a measurable set, and assume that $Y(E)$. We seek to show that $\mu(E) = \infty$. Since $\mu(E) = \inf_{\text{open } O \supset E} \mu(O)$, it suffices to show that for all open sets $O \supset E$, $\mu(O) = \infty$. Let O be such an open set. For all $y \in Y(E)$, $O^y \supset E^y \neq \emptyset$. Thus O^y is non empty and open, thus $m(O^y) > 0$.

   By inner regularity of the Lebesgue measure, there exists a compact $K^y \subset O^y$ such that $m(K^y) \geq m(O^y)/2$. By Urysohn's Lemma, there exists a continuous function $f^y \in C_c(\mathbb{R})$ such that $1_{K^y} \leq f^y \leq 1_{O^y}$.

Now for any finite subset $Y \subset Y(E)$, define the function

$$\begin{aligned} f^Y &: \mathbb{R} \times \mathbb{R}' \to \mathbb{R} \\ &(x, x') \mapsto \sum_{y \in Y} f^y(x) 1_{\{y\}}(x') \end{aligned}$$

Since $f^Y \prec O$, we have

$$\begin{aligned} \mu(O) &\geq \ell(f^Y) \\ &= \sum_{y \in Y} \int_{\mathbb{R}} f^y d\mu \\ &\geq \sum_{y \in Y} \int_{\mathbb{R}} 1_{K^y} d\mu && \text{since } 1_{K^y} \leq f_y \\ &= \sum_{y \in Y} m(K^y) \\ &\geq \frac{1}{2} \sum_{y \in Y} m(O^y) \end{aligned}$$

therefore $\mu(O) \geq \frac{1}{2} \sup_{\text{finite } Y \subset Y(E)} \sum_{y \in Y} m(O^y)$, which is the sum of uncountably many strictly positive elements, thus infinite.[2] This shows that $\mu(O) = \infty$.

3. Consider the Borel set $E = \{0\} \times \mathbb{R}'$. Show that $\mu(E) = \infty$, but $\mu(K) = 0$ for every compact set $K \subset E$. Therefore μ is not inner regular on arbitrary Borel sets.

proof We have $Y(E) = \{y \in \mathbb{R}' : E^y \neq \emptyset\} = \mathbb{R}'$ is uncountable, thus by (b), $\mu(E) = \infty$. Now let K be a compact subset of E. Then by (a), $Y(K) = \{y \in \mathbb{R}' : K^y \neq \emptyset\}$ is finite, and $K = \{0\} \times \{y_1, \ldots, y_k\}$. For all $\epsilon > 0$, $K \subset O_\epsilon$, where $O_\epsilon = (-\epsilon, \epsilon) \times \{y_1, \ldots, y_k\}$ is open. Thus

$$\begin{aligned} \mu(K) &\leq \mu(O_\epsilon) \\ &= \sup_{f \prec O_\epsilon} \ell(f) \\ &= \sup_{f \prec O_\epsilon} \sum_{i=1}^{k} \int_{\mathbb{R}} f^{y_i} d\mu \\ &\leq \sum_{i=1}^{k} \int_{\mathbb{R}} 1_{O_\epsilon}^{y_i} d\mu && \text{since } f \leq 1_{O_\epsilon} \\ &\leq 2k\epsilon \end{aligned}$$

since this holds for arbitrary $\epsilon > 0$, we have $\mu(K) = 0$

[2] if $(a_\alpha)_{\alpha \in A}$ is a collection of uncountably many positive reals, then $\sup_{\text{finite } B \subset A} \sum_{\alpha \in B} x_\alpha = \infty$. Indeed, let $A_n = \{\alpha : x_\alpha > 1/n\}$. Since $A \cup_{n \in \mathbb{N}} A_n$, there exists n such that A_n is uncountable, and the result follows.

(7.5) Let X be a locally compact Hausdorff space. Show that any positive linear functional $\ell : C_0(X) \to \mathbb{R}$ is bounded.

proof Suppose by contradiction that ℓ is not bounded. Then for all $M > 0$, there exists $f \in C_0(X)$ such that $\ell(f) > M$ and $\|f\| = 1$(where the norm here is $\|f\| = \sup_{x\in X} |f(x)|$)

Construct a sequence (f_n) of elements of $C_0(X)$ as follows: for all n, let $f_n \in C_0(X)$ such that $\|f_n\| = 1$ and $\ell(f_n) > 2^n$. Then consider the function

$$\begin{aligned} f : X &\to \mathbb{R} \\ x &\mapsto f(x) = \sum_{n\in\mathbb{N}} 2^{-n}|f_n(x)| \end{aligned}$$

(this is well defined since $|f_n(x)| \leq 1$ for all x). We have

- f is the uniform limit of the sequence $F_n = \sum_{k\leq n} 2^{-k}|f_k|$ (indeed, for all $\epsilon > 0$, there exists N such that $\sum_{k\geq N} 2^{-n} \leq \epsilon$, thus for all $n \geq N$ and for all x

$$|f(x) - F_n(x)| = \sum_{k\geq n+1} 2^{-n}|f_n(x)| \leq \sum_{k\geq n+1} 2^{-n} \leq \sum_{k\geq N} 2^{-n} \leq \epsilon$$

  Therefore f is the uniform limit of the continuous functions (F_n), and is continuous.

- $f \in C_0(X)$: indeed, let $\epsilon > 0$. For all $n \in \mathbb{N}$, $f_n \in C_0(X)$, thus there exists a compact K_n such that for all $x \in K_n^c$, $|f_n(x)| \leq \epsilon/2$. Consider the compact set $K = \cap_{n\in\mathbb{N}} K_n$. We have for all $x \in K^c$,

$$|f(x)| = \sum_{n\in\mathbb{N}} 2^-n|f_n(x)| \leq \frac{1}{2}\epsilon \sum_{n\in\mathbb{N}} 2^{-n} = \epsilon$$

- norm of f: since $\|f_n\| = 1$ for all n, we have for all $x \in X$, $|f(x)| = \sum_{n\in\mathbb{N}} 2^{-n}|f_n(x)| \leq \sum_{n\in\mathbb{N}} 2^{-n} = 2$, thus

$$\|f\| \leq 2$$

- finally, for all n, $f \geq F_n \geq 0$, thus

$$\begin{aligned} \ell(f) &\geq \ell(F_n) \\ &= \ell(\sum_{k\leq n} 2^{-n}|f_n|) \\ &= \sum_{k\leq n} 2^{-n}\ell(|f_n|) && \text{by linearity} \\ &\geq \sum_{k\leq n} 2^{-n}2^n && \text{since } \ell(|f_n|) \geq \ell(f_n) = 2^n \\ &\geq n \end{aligned}$$

  which contradicts the fact that $\ell(f)$ is finite.

(7.6) Let $k \in \mathbb{N}$. Define $C^k([0,1])$ to be the set of all functions $f \in C_0([0,1])$ which are k times continuously differentiable. This means that the one-sided derivatives exist at the endpoints and that derivatives of orders $0 \le n \le k$ are continuous on $[0,1]$, provided that they are defined to be appropriate one-sided derivatives at the end points. $C^k = C^k([0,1])$ is a normed vector space with norm

$$\|f\|_{C^k} = \sum_{j=0}^{k} \max_{x\in[0,1]} |f^{(j)}(x)|$$

where $f^{(j)}$ is the $j-$th derivative, and $f^{(0)} = f$. Show that every $\ell \in (C^k)^*$ takes the form

$$\ell(f) = \int_{[0,1]} f^{(k)} d\mu + \sum_{j=0}^{k-1} c_j f^{(j)}(0)$$

for some Borel measure μ on $[0,1]$ and some constants $c_j \in \mathbb{C}$, uniquely determined by ℓ.

proof Consider the linear function

$$\begin{aligned} \phi_k : C([0,1]) &\to C^k([0,1]) \\ g &\mapsto \phi(g) \end{aligned}$$

such that $\phi(g)$ is the unique function such that

$$\begin{aligned} &\phi_k(g)^{(k)} = g \\ &\phi_k(g)^{(j)}(0) = 0 \qquad\qquad \forall j \in \{0, \ldots, k-1\} \end{aligned}$$

more precisely, ϕ_k is defined by induction on k by $\phi_0 = id$, and

$$\phi_k(g)(x) = \int_{[0,x]} \phi_{k-1}(g)(u)du$$

(then $\phi_k(g)(0) = 0$, and for all $j \in \{1, \ldots, k\}$, $\phi_k(g)^{(j)} = \phi_{k-1}(g)^{(j-1)}$)
consider the linear functional

$$\begin{aligned} \tilde{\ell} : C([0,1]) &\to \mathbb{C} \\ f &\mapsto \tilde{\ell}(g) = \ell(\phi_k(g)) \end{aligned}$$

By the Riesz Representation theorem, there exists a Borel measure μ such that for all $g \in C([0,1])$

$$\tilde{\ell}(g) = \int_{[0,1]} g d\mu$$

therefore we have for all $f \in C^k([0,1])$, $f^{(k)} \in C([0,1])$, and we can apply $\tilde{\ell}$ to obtain

$$\ell(\phi(f^{(k)})) = \tilde{\ell}(f^{(k)}) = \int_{[0,1]} f^{(k)} d\mu$$

Next, we prove by induction on k, for all $f \in C^k([0,1])$, and for all $x \in [0,1]$

$$\phi_k(f^{(k)})(x) = f(x) - \sum_{j=0}^{k-1} x^j \frac{f^{(j)}(0)}{j!}$$

The property is true for $k = 0$ (by definition of ϕ_0). Now assume it is true for $k-1$, then we have for all $f \in C^k([0,1])$, by definition of ϕ_k,

$$\begin{aligned}
\phi_k(f^{(k)})(x) &= \int_{[0,x]} \phi_{k-1}(f^{(k)})(u)du \\
&= \int_{[0,x]} f'(u) - \sum_{j=0}^{k-2} u^j \frac{f^{(j+1)}(0)}{j!} du \quad \text{by writing } f^{(k)} = (f')^{(k-1)} \text{ and applying the induction hypothesis} \\
&= f(x) - f(0) - \sum_{j=0}^{k-2} x^{j+1} \frac{f^{(j+1)}(0)}{(j+1)!} \\
&= f(x) - \sum_{i=0}^{k-1} x^i \frac{f^{(i)}(0)}{i!}
\end{aligned}$$

which completes the induction. Finally, using the expression of $\phi_k(f^{(k)})$, we conclude

$$\int_{[0,1]} f^{(k)} d\mu = \ell(\phi(f^{(k)})) = \ell(f) - \sum_{j=0}^{k-1} \frac{\ell(x^j)}{j!} f^{(j)}(0)$$

i.e.

$$\ell(f) = \int_{[0,1]} f^{(k)} d\mu + \sum_{j=0}^{k-1} \frac{\ell(x^j)}{j!} f^{(j)}(0)$$

where μ is the unique Borel measure given by the Riesz Representation theorem, and the constants $c_j = \frac{\ell(x_j)}{j!}$ are determined by ℓ.