MATH 202B - Problem Set 3

Walid Krichene (23265217)

February 13, 2013

(3.1) The parallelogram law for L^2 norms states that for all $f, g \in L^2$,

$$\|f+g\|_2^2 + \|f-g\|_2^2 = 2\|f\|_2^2 + 2\|g\|_2^2$$

which can be viewed as a refinement of the triangle inequality $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$ by writing

$$\begin{aligned}\|f+g\|_2^2 &\leq 2\|f\|_2^2 + 2\|g\|_2^2 - \|f-g\|_2^2 \\ &= (\|f\|_2 + \|g\|_2)^2 + (\|f\|_2 - \|g\|_2)^2 - \|f-g\|_2^2\end{aligned}$$

thus if $\|f\|_2 = \|g\|_2$ and $\|f+g\|_2 = \|f\|_2 + \|g\|_2$, then $f = g$ a.e.

(3.1.a) show that more generally, if $\|f+g\|_2 = \|f\|_2 + \|g\|_2$, then there exists $c \geq 0$ such one function is c times the other almost everywhere.

answer First, observe that the case $\|g\|_2 = 0$ is trivial since it implies $g = 0$ almost everywhere and the result is true using the constant $c = 0$.

Now assume $\|g\|_2 > 0$. First, we have

$$\begin{aligned}\|f+g\|_2^2 &= \int_X (f+g)^2 d\mu \\ &= \int_X f^2 + \int_X g^2 + 2\int_X fg d\mu \\ &= \|f\|_2^2 + \|g\|_2^2 + 2\int_X fg d\mu \\ &= (\|f\|_2 + \|g\|_2)^2 + 2\left(\int_X fg d\mu - \|f\|_2\|g\|_2\right)\end{aligned}$$

therefore we have $\|f+g\|_2 = \|f\|_2 + \|g\|_2$ if and only if $\int_X fg d\mu = \|f\|_2\|g\|_2$.

Now consider the function

$$\begin{aligned}\phi : \mathbb{R} &\to \mathbb{R} \\ t &\mapsto \phi(t) = \|f+tg\|_2^2\end{aligned}$$

. We have similarly

$$\phi(t) = \|f\|_2^2 + t^2\|g\|_2^2 + 2t\int_X fg d\mu$$

which is a polynomial of degree 2 in t. The discriminant is

$$\Delta = \left(2\int_X fg d\mu\right)^2 - 4\|g\|_2^2\|f\|_2^2 = 0$$

therefore ϕ has a real root, i.e. there exists $t_0 \in \mathbb{R}$ such that $\phi(t) = \|f + t_0 g\|_2^2 = 0$, i.e.

$$\int_X (f(x) + t_0 g(x))^2 d\mu(x) = 0$$

therefore we have for almost every x, $f(x) = cg(x)$ where $c = -t_0$. Finally, we can rewrite the condition $\|f + g\|_2 = \|f\|_2 + \|g\|_2$ as

$$\|(1 + c)g\|_2 = (|c| + 1)\|g\|_2$$

i.e.

$$|1 + c| = |c| + 1$$

(since $\|g\|_2 > 0$). This requires in particular that $c \geq 0$.

(3.1.b) Counter example: for $p \in \{1, \infty\}$, find functions f, g such that $\|f + g\|_p = \|f\|_1 + \|g\|_p$, yet neither is a constant multiple of the other almost everywhere.

answer Consider the measure space $(\mathbb{R}, \mathcal{B}, \mu)$.

- Counter example for $p = 1$: consider the functions

$$f = 1_{[0,1]}$$
$$g = 1_{[1,2]}$$

  We have

$$\|f\|_1 = \|g\|_1 = 1$$
$$\|f + g\|_1 = 2$$

  therefore we have $\|f + g\|_1 = \|f\|_1 + \|g\|_1$ yet the functions are not proportional almost everywhere.

- Counter example for $p = \infty$: consider the functions

$$f = 1_{[0,1]}$$
$$g = 1_{[0,2]}$$

  We have

$$\|f\|_\infty = \|g\|_\infty = 1$$
$$\|f + g\|_\infty = 2$$

  therefore we have $\|f + g\|_\infty = \|f\|_\infty + \|g\|_\infty$ yet the functions are not proportional almost everywhere.

(3.2) All functions are assumed real valued. Clarkson's inequalities extend the previous discussion to other exponents, in a slightly weaker form.

(3.2.a) Let $p \in [2, \infty)$. For any $f, g \in L^p$, we have

$$\|f+g\|_p^p + \|f-g\|_p^p \leq 2^{p-1}(\|f\|_p^p + \|g\|_p^p)$$

proof we first show that for reals $a \geq b \geq 0$, we have

$$(a+b)^p + (a-b)^p \leq 2^{p-1}(a^p + b^p) \tag{1}$$

consider the function

$$\begin{aligned}\phi : [0,1] &\to \mathbb{R} \\ t &\mapsto \phi(t) = (a+tb)^p + (a-tb)^p - 2^{p-1}(a^p + (tb)^p)\end{aligned}$$

showing the desired inequality is equivalent to showing that $\phi(1) \leq 0$. And since we have

$$\phi(0) = (2 - 2^{p-1})a^p \leq 0$$

it suffices to show that for all $t \in (0,1]$, $\phi'(t) \leq 0$. We have

$$\begin{aligned}\phi'(t) &= pb(a+tb)^{p-1} - pb(a-tb)^{p-1} - 2^{p-1}pb(tb)^{p-1} \\ &= pb(tb)^{p-1}\left((\frac{a}{tb}+1)^{p-1} - (\frac{a}{tb}-1)^{p-1} - 2^{p-1}\right) \\ &= pb(tb)^{p-1}\psi(t)\end{aligned}$$

where $\psi(t) = (\frac{a}{tb}+1)^{p-1} - (\frac{a}{tb}-1)^{p-1} - 2^{p-1}$. We have for $t \in (0,1]$

$$\psi'(t) = -\frac{1}{bt^2}(p-1)\left((\frac{a}{tb}+1)^{p-2} - (\frac{a}{tb}-1)^{p-2}\right)$$

but since $t \leq 1$ and $a \geq b$, we have $\frac{1}{t} \geq 1$ and $\frac{a}{b} \geq 1$, thus

$$\frac{a}{tb} - 1 \geq 0$$

and it follows that

$$\frac{a}{tb} + 1 \geq \frac{a}{tb} - 1 \geq 0$$

and using the fact that $u \mapsto u^{p-2}$ is increasing on $[0, \infty)$ (since $p - 2 \geq 0$), we have for all $t \in (0,1]$

$$\psi'(t) \leq 0$$

and since $\lim_{t\to 0} \psi(t) = 0$, we have for all $t \in (0,1]$, $\psi(t) \leq 0$. Finally, from the expression of $\phi'(t)$, we conclude that for all $t \in (0,1]$, $\phi'(t) \leq 0$, therefore ϕ is non-increasing on $(0,1]$ and we obtain the desired result.

Next, it follows from inequality (1) that for all real numbers a, b,

$$|a+b|^p + |a-b|^p \leq 2^{p-1}(|a|^p + |b|^p) \tag{2}$$

(it suffices to apply the previous inequaity)

Now consider the functions f, g. We have

$$\begin{aligned}\|f+g\|_p^p + \|f-g\|_p^p &= \int_X |f(x)+g(x)|^p + |f(x)-g(x)|^p d\mu(x) \\ &\leq \int_X 2^{p-1}(|f(x)|^p + |g(x)|^p) d\mu(x) && \text{using inequality (2)} \\ &= 2^{p-1}(\|f\|_p^p + \|g\|_p^p)\end{aligned}$$

which proves the desired result.

(3.2.b) Show that as a consequence, if $\|f\|_p = \|g\|_p = 1$, then $\|f+g\|_p \le 2(1-2^{-p}\|f-g\|_p^p)^{1/p}$

proof If $\|f\|_p = \|g\|_p = 1$, then inequality (2) becomes

$$\|f+g\|_p^p + \|f-g\|_p^p \le 2^p$$

i.e.

$$\begin{aligned}\|f+g\|_p &\le (2^p - \|f-g\|_p^p)^{1/p}\\ &= 2(1-2^{-p}\|f-g\|_p^p)^{1/p}\end{aligned}$$

(3.2.c) Show that if $f, g \in L^p$ satisfy $\|f+g\|_p = \|f\|_p + \|g\|_p$, then there exists c such that one function is c times the other almost everywhere.

proof First, if $\|f\|_p = 0$, then f is zero almost everywhere, and the result holds with the constant $c = 0$. Similarly, the result holds wen $\|g\|_p = 0$.

Now assume without loss of generality, that $\|f\|_p \ge \|g\|_p > 0$. Let $c = \|g\|_p/\|f\|_p$. Then we have

$$\|cf\|_p = \|g\|_p$$

Now observe that if $\|f+g\|_p = \|f\|_p + \|g\|_p$, then we have[1]

$$\|cf+g\|_p = c\|f\|_p + \|g\|_p = 2\|g\|_p$$

now consider the functions $\tilde{f} = f/\|f\|_p$ and $\tilde{g} = -g/\|g\|_p$. We have

$$\begin{aligned}\|\tilde{f}+\tilde{g}\|_p &= \|cf/\|g\|_p + g/\|g\|_p\|_p\\ &= \|cf+g\|_p/\|g\|_p\\ &= 2\end{aligned}$$

And since $\tilde{f}$ and $\tilde{g}$ have unit norm, we can apply the result of (2.b), and we obtain

$$\begin{aligned}\|\tilde{f}-\tilde{g}\|_p &\le 2(1-2^-p\|\tilde{f}+\tilde{g}\|_p^p)^{1/p}\\ &= 2(1-2^-p2^p)^{1/p}\\ &= 0\end{aligned}$$

therefore $\|\tilde{f}-\tilde{g}\|_p = 0$, and $\tilde{f} = \tilde{g}$ almost everywhere. In other words, $g = \frac{\|g\|_p}{\|f\|_p} f$ almost everywhere.

[1] indeed, we have

$$\begin{aligned}\|cf+g\|_p &= \|f+g+(1-c)f\|_p\\ &\ge \|f+g\|_p - \|(1-c)f\|_p && \text{by the triangle inequality}\\ &= \|f\|_p + \|g\|_p - (1-c)\|f\|_p && \text{since } c \in [0,1]\\ &= c\|f\|_p + \|g\|_p\end{aligned}$$

the reverse inequality $\|cf+g\|_p \le c\|f\|_p + \|g\|_p$ follows simply from the triangle inequality.

(3.3) Let $p, q \in (1, \infty)$ be a pair of conjugate exponents. Let $f \in L^p$ and $g \in L^q$ be $\mathbb{C}$-valued functions with strictly positive norms. Suppose that

$$|\int_X f\bar{g}d\mu| = \|f\|_p \|g\|_q$$

(3.3.a) Suppose that f, g are real-valued, non-negative, then there exists $c > 0$ such that $g = cf^{p/q}$ almost everywhere.

proof First, we can rewrite the equality as

$$\int_X \frac{f}{\|f\|_p} \frac{g}{\|g\|_q} d\mu = 1$$

i.e.

$$\int_X \tilde{f}\tilde{g}d\mu = 1$$

where

$$\tilde{f} = \frac{f}{\|f\|_p}$$
$$\tilde{g} = \frac{g}{\|g\|_q}$$

Now define the function

$$\phi : X \to \mathbb{C}$$
$$x \mapsto \phi(x) = \tilde{f}(x)\tilde{g}(x) - \frac{1}{p}\tilde{f}(x)^p - \frac{1}{q}\tilde{g}(x)^q$$

we have by Holder's inequality, for all $x \in X$, $\phi(x) \leq 0$. We also have

$$\begin{aligned}\int_X \phi d\mu &= \int_X \tilde{f}\tilde{g}d\mu - \frac{1}{p} - \frac{1}{q} \\ &= 1 - \frac{1}{p} - \frac{1}{q} \\ &= 0\end{aligned}$$

therefore we must have $\phi = 0$ almost everywhere, i.e. for almost every x,

$$\tilde{f}(x)\tilde{g}(x) = \frac{1}{p}\tilde{f}(x)^p + \frac{1}{q}\tilde{g}(x)^q$$

this corresponds to having equality in Young's inequality for the exponential function, applied between the points $p \ln \tilde{f}(x)$ and $q \ln \tilde{g}(x)$, with coefficients $1/p$ and $1/q$: indeed, the above equation can be rewritten

$$e^{\frac{1}{p}p\ln(\tilde{f}(x)) + \frac{1}{q}\ln\tilde{g}(x)} = \frac{1}{p}e^{p\ln\tilde{f}(x)} + \frac{1}{q}e^{q\ln\tilde{g}(x)}$$

by strict convexity of the exponential, equality only occurs if $1/p = 0$, or $1/p = 1$, or $p \ln \tilde{f}(x) = q \ln \tilde{g}(x)$. Since the first two cases cannot occur, we have for almost every x

$$p \ln \tilde{f}(x) = q \ln \tilde{g}(x)$$

i.e.

$$\tilde{g}(x) = \tilde{f}(x)^{p/q}$$

or

$$g(x) = \frac{\|g\|_q}{\|f\|_p^{p/q}} f(x)^{p/q}$$

(3.3.b) In the general case, show that there exists $c \in \mathbb{C}$ such that $g = cf^{p/q}$, i.e. there exist $r \geq 0$ and $\theta \in [0, 2\pi]$ such that for almost every x,

$$|g(x)| = r|f(x)|^{p/q}$$
$$\arg(g(x)) = \theta + \arg(f(x)) \mod (2\pi)$$

First, we observe that

$$\begin{aligned} |\int_X f\bar{g}d\mu| &\leq \int_X |f||g|d\mu \\ &\leq \|f\|_p\|g\|_q d\mu & \text{by Holder's inequality} \end{aligned}$$

But since we have $|\int_X f\bar{g}d\mu| = \|f\|_p\|g\|_q$, we must have equality in the previous inequalities. In particular,

$$\int_X |f||g|d\mu = \leq \|f\|_p\|g\|_q d\mu$$

applying (3.b) to the real-valued functions $x \mapsto |g(x)|$ and $x \mapsto |f(x)|$, there exists $r \geq 0$ such that $|g| = r|f|^{p/q}$.

Now to prove the result for the arguments, we need to use the fact that equality holds in $|\int_X f\bar{g}| \leq \int_X |f||g|d\mu$. Let us write $\int_X f\bar{g}d\mu = \alpha e^{i\phi}$, where $\alpha \geq 0$, and $\gamma \in [0, 2\pi]$. Then we have

$$\begin{aligned} |\int_X f\bar{g}d\mu| &= \alpha = \Re(\alpha) \\ &= \Re(\int_X f\bar{g}d\mu e^{-i\gamma}) \\ &= \int_X \Re(f\bar{g}e^{-i\gamma})d\mu \end{aligned}$$

now let

$$\begin{aligned} \phi &: X \to \mathbb{R} \\ x &\mapsto \phi(x) = \Re(f(x)\bar{g}(x)e^{-i\gamma}) - |f(x)||g(x)| \end{aligned}$$

we have for all x, $\phi(x) \leq 0$ (the real part of a complex number is no greater than its modulus). But we also have

$$\begin{aligned} \int_X \phi d\mu &= \int_X \Re(f\bar{g}e^{-i\gamma})d\mu - \int_X |f||g| \\ &= |\int_X f\bar{g}d\mu| - \int_X |f||g| \\ &= 0 \end{aligned}$$

therefore we have $\phi(x) = 0$ almost everywhere, i.e. for almost every x

$$\Re(f(x)\bar{g}(x)e^{-i\gamma}) = |f(x)||g(x)|$$

i.e. $f(x)\bar{g}(x)e^{-i\gamma}$ is real non-negative (since equal to its modulus). Therefore for almost every x,

$$\arg(f(x)\bar{g}(x)e^{-i\gamma}) = 0 \mod (2\pi)$$

i.e.

$$\arg(f(x)) - \arg(g(x)) - \gamma = 0 \mod (2\pi)$$

this concludes the proof.

(3.4) Consider the measure space $(\mathcal{R}, \mathcal{A}, \mu)$, where

- $\mathcal{R} = \times_{i=1}^{n} [a_i, b_i]$
- $\mathcal{A}$ is the set of all Borel subsets of $\mathcal{R}$
- μ is the n dimensional Lebesgue measure.

A function $f : \mathcal{R} \to \mathbb{C}$ is said to be infinitely differentiable if f is continuous, f is infinitely differentiable on the $\text{int}(\mathcal{R})$, and each derivative extends continuously to the boundary of $\mathcal{R}^2$.

Show that for all $p \in [1, \infty)$, $C^\infty(\mathcal{R})$ is dense in $L^p(\mathcal{R})$ (in the $\|.\|_p$ sense).

proof Let $f \in L^p(\mathcal{R})$, and let $\epsilon > 0$. By definition, we can approach f from below by a non-decreasing sequence of simple functions (s_n). Then we have sequence $|s_n - f|^p$ converges pointwise to 0, and is dominated by $2|f|^p$, which is integrable, therefore by the DCT

$$\lim_n \int_X |s_n - f|^p d\mu = 0$$

i.e. $(\|s_n - f\|_p)_n$ converges to 0. Therefore there exists a simple function s such that $\|f - s\|_p \leq \epsilon/3$. Next, we cn approximate the simple function s by a continuous function $\bar{s}$: let the canonical expansion of s be given by

$$s = \sum_{i=1}^{n} c_i 1_{E_i}$$

where for all i, $c_i \in \mathbb{R}$ and E_i is a Borel set of $\mathcal{R}$ (the E_is are disjoint). Then we can approximate, for all i, $c_i 1_{E_i}$ by a continuous function $\bar{s}_i$ (using for example Urysohn's Lemma). Let $\bar{s}_i$ be a continuous function such that $\|c_i 1_{E_i} - \bar{s}_i\|_p \leq \epsilon/(2n)$, and let $\bar{s} = \sum_{i=1}^{n} \bar{s}_i$. Then we have

$$\begin{aligned}
\|s - \bar{s}\|_p &= \|\sum_{i=1}^{n} c_i 1_{E_i} - \bar{s}_i\|_p \\
&\leq \sum_{i=1}^{n} \|c_i 1_{E_i} - \bar{s}_i\|_p \\
&\leq \epsilon/3
\end{aligned}$$

Finally, using the Stone Weirstrass theorem on the compact space $\mathcal{R}$, we have that the Algebra of complex valued polynomial functions $P(\mathcal{R}, \mathbb{C})$ is dense in the space of continuous functions (wrt the uniform convergence metric $\|g\|_\infty = \sup_{x \in \mathcal{R}} |g(x)|$).

Indeed, this algebra contains constants, separates points, and is closed under multiplication, addition, scalar multiplication and conjugation.

Therefore there exists a polynomial $p \in P(\mathcal{R})$ such that $\sup_{x \in \mathcal{R}} |p(x) - \bar{s}(x)| \leq \frac{\epsilon}{3\mu(\mathcal{R})^{1/p}}$, and we have

$$\begin{aligned}
\|p - \bar{s}\|_p &= \left(\int_X |p - \bar{s}|^p d\mu \right)^{1/p} \\
&\leq \frac{\epsilon}{3} \left(\int_X \frac{1}{\mu(\mathcal{R})} d\mu \right)^{1/p} \\
&= \frac{\epsilon}{3}
\end{aligned}$$

combining the three bounds, we obtain

$$\|f - p\|_p \leq \|f - s\|_p + \|s - \bar{s}\|_p + \|\bar{s} - p\|_p \leq \epsilon$$

which proves that $P(\mathcal{R}, \mathbb{C})$ is dense in $L^p(\mathcal{R})$. And since the space of polynomials on $\mathcal{R}$ is a subset of $C^\infty(\mathcal{R})$, this proves the desired result.

(3.5) Let $p \in [1, \infty)$. Suppose that

- $f_n \in L^p$
- $f_n \to f$ almost everywhere
- there exists $M < \infty$ such that for all n, $\|f_n\|_p \leq M < \infty$

Show that if $(\|f_n\|_p)_n \to \|f\|_p$, then $(\|f_n - f\|_p)_n \to 0$.

proof First, we show that for all $A \in \mathcal{A}$,

$$\int_A |f_n|^p d\mu \to \int_A |f|^p d\mu$$

using Fatou's lemma on A and $X \setminus A$, we have

$$\int_A |f|^p = \int_A \liminf_n |f_n|^p \leq \liminf_n \int_A |f_n|^p$$
$$\int_{X \setminus A} |f|^p = \int_{X \setminus A} \liminf_n |f_n|^p \leq \liminf_n \int_{X \setminus A} |f_n|^p$$

next, we rewrite the second inequality (using the fact that $\liminf_n -u_n = -\limsup_n u_n$)

$$-\int_{X \setminus A} |f|^p \geq \limsup_n -\int_{X \setminus A} |f_n|^p$$

and since $\int_X |f|^p = \lim_n \int_X |f_n|_p = \limsup_n \int_X |f_n|_p$ and is finite, we have

$$\begin{aligned}
\int_A |f|^p d\mu &= \int_X |f|^p d\mu - \int_{X \setminus A} |f|^p d\mu \\
&\geq \limsup_n \int_X |f_n|_p + \limsup_n -\int_{X \setminus A} |f_n|^p \\
&\geq \limsup_n \left(\int_X |f_n|_p - \int_{X \setminus A} |f_n|^p\right) \qquad \text{using } \sup u_n = \sup v_n \geq \sup(u_n + v_n) \\
&= \limsup_n \int_A |f_n|_p
\end{aligned}$$

Therefore we have

$$\limsup_n \int_A |f_n|_p \leq \int_A |f|^p d\mu \leq \liminf_n \int_A |f_n|^p$$

but the last term is a lower bound on the first term, (the lim inf is always no greater than the lim sup), therefore we have equality, and the claim is proved.

Proof of main result: let $\epsilon > 0$ be given. Let $A = \{x \in X : |f(x)| > 0\}$. A is measurable. First, we decompose the integral into

$$\int_X |f_n - f|^p d\mu = \int_A |f_n - f|^p d\mu + \int_{X \setminus A} |f_n|^p d\mu$$

using the fact that for all $x \in X \setminus A$, $f(x) = 0$. Then by the previous claim, $\int_{X\setminus A} |f_n|^p d\mu$ converges to $\int_{X\setminus A} |f|^p = 0$. Therefore there exists N_1 such that

$$\forall n \geq N_1,\ \int_{X\setminus A} |f_n|^p \leq \epsilon/4 \tag{3}$$

now define a new measure ν on $A \cap \mathcal{A}$ as follows:

$$\forall E \in A \cap \mathcal{A},\ \nu(E) = \int_E |f|^p d\mu$$

we have $\nu(A) = \int_A |f|^p d\mu \leq M < \infty$. Therefore $(A, A\cap\mathcal{A}, \nu)$ is a finite measure space and we can apply Egoroff's theorem: we have (f_n/f) converges to 1 almost everywhere on A (note that we had to restrict this discussion to the set A so that f_n/f be defined). Thus by Egoroff's theorem, there exists a measurable set $F \in A \cap \mathcal{A}$ such that

- $\nu(F) \leq \epsilon/8$ (we assume $\nu(A) > 0$, otherwise f is 0 almost everywhere, in which case the result is trivial)
- (f_n/f) converges uniformly to 1 on $A \setminus F$.

Now we seek to bound the terms $\int_F |f_n - f|^p d\mu$ and $\int_{A\setminus F} |f_n - f|^p d\mu$. First, we have

$$\int_F |f_n - f|^p d\mu \leq \int_F |f_n|^p d\mu + \int_F |f|^p d\mu$$

and since $\int_F |f_n|^p d\mu$ converges to $\int_F |f|^p d\mu$, there exists N_2 such that

$$\begin{aligned}\forall n \geq N_2,\ \int_F |f_n - f|^p d\mu &\leq 2\int_F |f|^p d\mu + \epsilon/4 \\ &= 2\nu(F) + \epsilon/4 \\ &\leq \epsilon/4 + \epsilon/4 \end{aligned} \tag{4}$$

By uniform convergence, there exists N_2 such that

$$\forall n \geq N_3,\ \forall x \in A\setminus E,\ |f_n(x)/f(x) - 1| \leq \frac{\epsilon}{4\nu(A)}$$

(we can assume $\nu(A) > 0$ because otherwise f is 0 a.e., in which case the result is trivial) thus

$$\begin{aligned}\forall n \geq N_3,\ \int_{A\setminus F} |f_n - f|^p d\mu &= \int_{A\setminus F} |f_n/f - 1|^p |f|^p d\mu \\ &\leq \int_{A\setminus F} \frac{\epsilon}{4\nu(A)} d\nu \\ &\leq \frac{\epsilon}{4\nu(A)} \nu(A\setminus F) \\ &\leq \epsilon/4 \end{aligned} \tag{5}$$

Finally, combining the above bounds (3) (4) (5), we have for all $n \geq \max(N_1, N_2, N_3)$

$$\begin{aligned}\int_X |f_n - f|^p d\mu &= \int_F |f_n - f|^p d\mu + \int_{A\setminus F} |f_n - f|^p d\mu + \int_{X\setminus A} |f_n| d\mu \\ &\leq \epsilon\end{aligned}$$

this completes the proof.

(3.6) Let $1 \leq p \leq r \leq q \leq \infty$. Let $\theta \in [0, 1]$ be the unique number such that

$$r^{-1} = \theta p^{-1} + (1 - \theta) q^{-1}$$

Show that for any measurable function f,

$$\|f\|_r \leq \|f\|_p^\theta \|f\|_q^{1-\theta}$$

conclude that $L^p \cap L^q \subset L^r$.

proof We consider the following cases:

1. $p = r = q = \infty$. Then the result becomes $\|f\|_\infty \leq \|f\|_\infty^\theta \|f\|_\infty^{1-\theta}$ and this trivially holds (with equality) for all $\theta \in [0, 1]$
2. $p < r = q = \infty$. Then we have $0 = \theta p^{-1} + (1 - \theta)0$, thus we must have $\theta = 0$. The result becomes $\|f\|_\infty \leq \theta \|f\|_\infty$ which trivially holds with equality.
3. $p \leq r < q = \infty$. Then we have $r^{-1} = \theta p^{-1}$, i.e. $\theta = p/r$. Let $M = \|f\|_\infty$ be the essential supremum (finite since $f \in L^\infty$). Then the set $B = \{x : |f(x)| > M\}$ has measure 0. Then we have for all $x \in X \setminus B$, $|f(x)|/M \leq 1$, thus $(|f(x)|/M)^{r-p} \leq 1$ (since $r - p \geq 0$). Integrating, we have

$$\int_X |f|^r d\mu \leq \int_X |f|^p d\mu M^{r-p}$$
$$i.e.\ (\int_X |f|^r d\mu)^{1/r} \leq ((\int_X |f|^p d\mu)^{1/p})^{p/r} M^{(r-p)/r}$$
$$i.e.\ \|f\|_r \leq (\|f\|_p)^{p/r} \|f\|_\infty^{(r-p)/r}$$

   and we conclude observing that $p/r = \theta$ and $(r - p)/r = 1 - \theta$.

4. $p \leq r \leq q < \infty$. Then let

$$a = \frac{p}{r\theta}$$
$$b = \frac{q}{r(1-\theta)}$$

   a and b are conjugate, since $1/a + a/b = r(\theta p^{-1} + (1 - \theta)q^{-1}) = 1$. Then apply Holder's inequality:

$$\begin{aligned}
\int_X |f|^r d\mu &= \int_X |f|^r d\mu \\
&= \int_X |f|^{p/a} |f|^{q/b} d\mu && \text{using } p/a + q/b = r\theta + r(1-\theta) = r \\
&\leq \||f|^{p/a}\|_a \||f|^{q/b}\|_b && \text{using Holder's inequality} \\
&= (\int_X (|f|^{p/a})^a d\mu)^{1/a} (\int_X (|f|^{q/b})^b d\mu)^{1/b} \\
&= (\int_X |f|^p d\mu)^{r\theta/p} (\int_X |f|^q d\mu)^{(1-\theta)r/q}
\end{aligned}$$

   raising to the power $1/r$, we obtain the desired result.

Conclusion: using the bound above, if $f \in L^p \cap L^q$, then $\|f\|_r$ is finite since it is bounded by the product of two bounded terms. Thus $f \in L^r$.