MATH 202B - Problem Set 10

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(10.1) Let $k \in \mathbb{N}$, and let $X = C^k([0,1]^d)$ to be the set of all functions $f \in C_0([0,1]^d)$ such that for all $\alpha \in \Delta_k$, $f^{(\alpha)} = \frac{\partial f}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$ exists and is continuous on $(0,1)^d$, and extends continuously on $[0,1]^d$. Here $\Delta_k = \{\alpha \in \{0, \dots, k\}^d : \sum_i \alpha_i \leq k\}$. The norm of the space is

$$\|f\|_{C^k} = \sum_{\alpha \in \Delta_k} \sup_{x \in [0,1]^d} |f^{(\alpha)}(x)|$$

Show that every $\ell \in (C^k)^*$ takes the form

$$\ell(f) = \sum_{\alpha \in \Delta_k} \int_{[0,1]^d} f^{(\alpha)} d\mu_\alpha$$

for some Borel measures μ_α on $[0,1]^d$, uniquely determined by ℓ.

proof Equip Δ_k with the discrete topology, and consider the set $K = [0,1]^d \times \Delta_k$. K is compact as the product of compact spaces. Let $C(K)$ the set of continuous functions on K.

Now consider the linear operator

$$\Phi : X \to C(K)$$
$$f \mapsto \Phi(f) : (x, \alpha) \mapsto f^{(\alpha)}(x)$$

we have

$$\|\Phi(f)\|_{C(K)} = \sup_{\alpha \in \Delta_k} \sup_{x \in [0,1]^d} |f^{(\alpha)}(x)|$$

which defines an equivalent norm to $\|\cdot\|_{C^k}$. Therefore $\phi(X)$ is a closed subspace of $C(K)$. Let $\Phi^\dagger : \Phi(X) \to X$ be the inverse of Φ. $\Phi^\dagger$ is linear bounded.

Now let $\ell \in (C^k)^*$. Define the linear functional

$$L : \Phi(X) \to \mathbb{R}$$
$$g \mapsto L(g) = \ell(\Phi^\dagger(g))$$

Then L is a positive linear functional on $\Phi(X)$, and since $\Phi(X)$ is a compact subset of compactly supported functions, by the Riesz representation theorem, there exists a Radon measure μ on K such that for all $g \in \Phi(X)$

$$L(g) = \int_{[0,1]^d \times \Delta_k} g d\mu$$

and we can partition K into $K = \sqcup_{\alpha \in \Delta_k} [0,1]^d \times \{\alpha\}$

$$L(g) = \sum_{\alpha \in \Delta_k} \int_{[0,1]^d \times \{\alpha\}} g d\mu$$

now for all $f \in C^k$, let $g = \Phi(f)$. Then

$$\begin{aligned}\ell(f) &= \ell(\Phi^\dagger(\Phi(f))) \\ &= L(\Phi(f)) \\ &= \sum_{\alpha\in\Delta_k} \int_{[0,1]^d\times\{\alpha\}} \Phi(f) d\mu \\ &= \sum_{\alpha\in\Delta_k} \int_{[0,1]^d} f^{(\alpha)}(x) d\mu_\alpha\end{aligned}$$

where $\mu_\alpha(E) = \mu(E \times \{\alpha\})$

(10.2) A normed linear space is said to be separable if it is separable as a metric space, that is, has countable dense subset. Let X be a normed linear space. Show that if X^* is separable, then so is X.

proof Suppose that X^* is separable, and let $(\ell_n)_{n\in\mathbb{N}}$ be a countable dense subset of X^*.

Now construct a candidate subset of X as follows: fix $\ell \in X^*$, then for all $n \in \mathbb{N}$, we have

$$\|\ell_n\|_{X^*} = \sup_{x\in X, \|x\|_X=1} |\ell_n(x)|$$

thus there exists $x_n \in X$ such that

$$\begin{aligned}&\|x_n\|_X = 1 \\ &|\ell_n(x_n)| \geq \frac{1}{2}\|\ell_n\|_{X^*}\end{aligned}$$

This defines a sequence of elements (x_n). Now let

$$V = \operatorname*{span}_{\mathbb{Q}+i\mathbb{Q}}(x_n)$$

be the set of finite linear combinations of the elements of (x_n), with rational coefficients. Then Y is countable (Y can be written as $V = \cup_{n\in\mathbb{N}} \cup_{r_1,\ldots,r_n\in\mathbb{Q}+i\mathbb{Q}} \sum_{i=1}^n r_i x_i$), and we show that $\text{cl}(V) = X$.

First, we observe that $\text{cl}(V)$ is a subspace. To prove this, let $\tilde{V} = \text{span}_{\mathbb{C}}(x_n)$. Then we have

- $\tilde{V} \subset V$, thus $\text{cl}(\tilde{V}) \subset \text{cl}(V)$
- by density of $\mathbb{Q} + i\mathbb{Q}$ in $\mathbb{C}$, $\tilde{V} \subset \text{cl}(V)$. Thus $\text{cl}(\tilde{V}) \subset \text{cl}(\text{cl}(V)) = \text{cl}(V)$

this proves $\text{cl}(V) = \text{cl}(\tilde{V})$, thus $\text{cl}(V)$ is a subspace.

Now assume by contradiction that $\text{cl}(V) \neq X$. Then by a corollary of the Hahn-Banach theorem, there exists a bounded linear functional $\ell \in X^*$ such that $\ell_{|\,\text{cl}(V)} \equiv 0$, and $\|\ell\|_{X^*} > 0$. By density of $(\ell_n)_n$ in X^*, for all $\epsilon > 0$, there exists n such that $\|\ell - \ell_n\|_{X^*} \leq \epsilon$. In particular for $\epsilon = \frac{1}{4}\|\ell\|_{X^*}$, there exists $n \in \mathbb{N}$ such that

$$\|\ell - \ell_n\|_{X^*} \leq \frac{1}{4}\|\ell\|_{X^*} \tag{1}$$

then we have

$$\begin{aligned}
|\ell_n(x_n)| &= |(\ell_n - \ell)(x_n)| && \text{since } \ell_{|\,\mathrm{cl}(V)} \equiv 0 \\
&\leq \|\ell_n - \ell\|_{X^*}\|x_n\|_X \\
&= \|\ell_n - \ell\|_{X^*} && \text{since } \|x_n\| = 1 \\
&\leq \frac{1}{4}\|\ell\|_{X^*} && \text{by (1)}
\end{aligned}$$

but

$$\begin{aligned}
|\ell_n(x_n)| &\geq \frac{1}{2}\|\ell_n\|_{X^*} && \text{by construction of } x_n \\
&\geq \frac{1}{2}(\|\ell\|_{X^*} - \|\ell_n - \ell\|_{X^*}) && \text{by the triangle inequality} \\
&\geq \frac{1}{2}\frac{3}{4}\|\ell\|_{X^*} && \text{by (1)}
\end{aligned}$$

which leads to the contadiction

$$0 < \frac{3}{8}\|\ell\|_{X^*} \leq \frac{1}{4}\|\ell\|_{X^*}$$

(10.3) Let $X = \ell^\infty$ be the Banach space of real-valued bounded sequences, equipped with the supremum norm. Let X_0 be the smallest closed vector subspace of X containing all sequence of the form $s_1, s_2 - s_1, s_3 - s_2, \dots$ such that $s \in \ell^\infty$. Prove that the sequence $e = (1, 1, \dots)$ is not in X_0

proof Let

$$\begin{aligned}
f : \ell^\infty &\to \ell^\infty \\
(s) &\mapsto f(s) = (s_1, s_2 - s_1, s_3 - s_2, \dots)
\end{aligned}$$

We have f is linear, therefore $f(\ell^\infty)$ is a vector space, and we have $X_0 = \mathrm{cl}(f(\ell^\infty))$. Let $s \in \ell^\infty$, and consider the sequence $f(s)$. We have

$$\sum_{n=1}^{N} f(s)_n = s_1 + (s_2 - s_1) + \dots + (s_n - s_{n-1}) = s_n$$

thus $|\sum_{n=1}^{N} f(s)_n| \leq \|s\|_\infty$, in particular, for all $\epsilon > 0$, there exists n such that $f(s)_n < \epsilon$ (for otherwise $\sum_{n=1}^{N} f(s)_n > n\epsilon$ would be unbounded). Therefore for all $\epsilon > 0$, there exists n such that $1 - f(s)_n > 1 - \epsilon$, thus $\|f(s) - e\|_\infty \geq 1 - \epsilon$. Since this holds for arbitrary $\epsilon > 0$, we have

$$\|f(s) - e\|_\infty \geq 1$$

therefore taking the sup over all sequences $(s_n) \in \ell^\infty$,

$$d(e, f(\ell_\infty)) = \sup_{(s)\in\ell^\infty} \|f(s) - e\|_\infty \geq 1$$

in fact we have

$$d(e, f(\ell_\infty)) = 1$$

since $d(e, f(\ell^\infty)) \leq \|e - f(0)\|_\infty = \|e\|_\infty = 1$.

It follows that $d(e, X_0) = 1$, therefore $e \notin X_0$.

(10.4) A Banach limit is defined to be any member $L \in X^*$ with $\|L\| = 1$, $L(e) = 1$ and $L(x_0) = 0$ for all $x_0 \in X_0$. Prove the existence of a Banach limit.

proof We have X_0 is a closed subspace such that $e \notin X_0$, and $d(e, X_0) = 1$. By a corollary of the Hahn Banach theorem, there exists $L \in X^*$ such that $L_{|X_0} = 0$, $L(e) = 1$, and $\|L\| = 1/d(e, X_0) = 1$.

(10.5) Let $L(s_n)$ denote the Banach limit of $s \in \ell^\infty$. Prove that

(a) $L(s_n) \geq 0$ if $s_n \geq 0$ for all n

proof Let $s \in \ell^\infty$ such that for all n, $s_n \geq 0$. If $s \equiv 0$, then $L(s) = 0$. Otherwise, we can consider the image by L of $\frac{s}{\|s\|_\infty}$,

$$\begin{aligned} L\left(\frac{s}{\|s\|_\infty}\right) &= L\left(\frac{s}{\|s\|_\infty} - e + e\right) \\ &= L\left(\frac{s}{\|s\|_\infty} - e\right) + 1 \qquad \text{by linearity and the fact that } L(e) = 1 \end{aligned}$$

now for all n, $0 \leq s_n/\|s\|_\infty \leq 1$, thus $\|\frac{s}{\|s\|_\infty} - e\|_\infty \leq 1$. Thus

$$\left|L\left(\frac{s}{\|s\|_\infty} - e\right)\right| \leq \|L\| \left\|\frac{s}{\|s\|_\infty} - e\right\|_\infty \leq 1$$

in particular, $L\left(\frac{s}{\|s\|_\infty} - e\right) \geq -1$, and we conclude that

$$L\left(\frac{s}{\|s\|_\infty}\right) \geq 0$$

and finally, $L(s) = \|s\|_\infty L(\frac{s}{\|s\|_\infty}) \geq 0$

(b) $L(s_{n+1}) = L(s_n)$ for all $s \in \ell^\infty$

proof Consider the left-shift operator

$$\begin{aligned} S : \ell^\infty &\to \ell^\infty \\ (s_1, s_2, \dots) &\mapsto S(s) = (s_2, s_3, \dots) \end{aligned}$$

Let $s \in \ell^\infty$, and let $\tilde{s} = S(s) - s_1 e = (s_2 - s_1, s_3 - s_1, \dots)$. Then $\tilde{s} \in \ell^\infty$. We have

$$\begin{aligned} S(s) - s &= (s_2 - s_1, s_3 - s_2, s_4 - s_3, \dots) \\ &= (\tilde{s}_1, \tilde{s}_2 - \tilde{s}_1, \tilde{s}_3 - \tilde{s}_2, \dots) \end{aligned}$$

thus $S(s) - s \in X_0$, therefore $L(S(s) - s) = 0$, i.e.

$$L(S(s)) = L(s)$$

which proves the result.

(c) $L(s_n) = 0$ if s is eventually 0

proof First, by (b) we have by induction on N, for all $N \geq 1$, for all $s \in \ell^\infty$

$$L(S^N(s)) = L(s)$$

Let $s \in \ell^\infty$, such that (s_n) is eventually 0. Then there exists N such that for all $n \geq N$, $s_n = 0$. Therefore $S^N(s) \equiv 0$, and we have

$$L(s) = L(S^N(s)) = L(0) = 0$$

(d) $L(s_n) \in [\liminf_n s_n, \limsup_n s_n]$

proof First, we have by (a) that for all $s, t \in \ell^\infty$, if $s \leq t$ (meaning for all n, $s_n \leq t_n$), then $L(s) \leq L(t)$ (since $(t_n - s_n)$ is a nonnegative sequence)

Let $(s_n) \in \ell^\infty$. We have for all n, $\inf_k s_k \leq s_n \leq \sup_k s_k$, i.e.

$$(\inf_k s_k)e \leq s \leq (\sup_k s_k)e$$

Therefore by the previous remark,

$$\inf_k s_k L(e) \leq L(s) \leq \sup_k s_k L(e)$$

and using the fact that $L(e) = 1$

$$\inf_k s_k \leq L(s) \leq \sup_k s_k \tag{2}$$

now we have

$$\limsup_n s_n = \inf_n \sup_{k \geq n} s_k = \inf_n \sup_k S^n(s)_k$$

Thus for all $\epsilon > 0$, there exists N such that $\sup_k S^N(s)_k \leq \limsup_n s_n + \epsilon$. Applying (2) to the sequence $S^N(s)$, we have

$$L(s) = L(S^N(s)) \leq \sup_k S^N(s)_k \leq \limsup_n s_n + \epsilon$$

since this holds for arbitrary $\epsilon > 0$, we have

$$L(s) \leq \limsup_n s_n$$

similarly, we have $L(s) \geq \liminf_n s_n$ (using the fact that $\liminf_n s_n = \sup_n \inf_k S^n(s)_k$)

(e) $L(s_n) = x$ if s converges to c.

proof Let (s_n) be a converging sequence with limit c. Then we have $\limsup_n s_n = \liminf_n s_n = c$, therefore by (d), $L(s) = c$.

(10.6) Let V be a closed subspace of a Banach space X. Suppose that for each $x \in X$, $d(x,V) \leq \frac{1}{2}\|x\|$. Then $V = X$.

proof Assume by contradiction that there exists $x \in X$ and $x \notin V$. We construct a sequence (x_n) of elements of V that converges to x. More precisely, the sequence will satisfy that for all n,

$$0 < \|x - x_n\| \leq (3/4)^n \|x\|$$

($\|x - x_n\|$ is positive since $x \notin V$).

- for $n = 1$, since $d(x,V) \leq \frac{1}{2}\|x\|$, there exists $x_1 \in V$ such that

$$\|x - x_1\| \leq \frac{1}{2}\|x\| + \frac{1}{4}\|x\| = (3/4)\|x\|$$

  this defines the first term x_1 in the sequence.

- assume that the term x_n is constructed, and that $0 < \|x - x_n\| \leq (3/4)^n\|x\|$. Since $d(x - x_n, V) \leq \frac{1}{2}\|x - x_n\|$, there exists $v_{n+1} \in V$ such that

$$\|x - x_n - v_{n+1}\| \leq \frac{1}{2}\|x - x_n\| + \frac{1}{4}\|x - x_n\| = (3/4)\|x - x_n\| \leq (3/4)^{n+1}\|x\|$$

  letting $x_{n+1} = x_n + v_{n+1}$, we have $x_{n+1} \in V$ since V is a subspace and both $x_n, v_{n+1} \in V$, and

$$\|x - x_{n+1}\| \leq (3/4)^{n+1}\|x\|$$

This proves that (x_n) converges to x, thus $x \in \mathrm{cl}(V) = V$, which contradicts the assumption.

(10.7) Let X be a Banach space.

(a) Show that any finite dimensional subspace of X is closed.

proof Let V be a finite dimensional subspace of X, and let $(e_1, \ldots, e_d)$ be a basis of V. Let $\|\cdot\|_{|V}$ be the restriction of the norm to the subspace V, and define the norm N as follows

$$N : V \to \mathbb{R}_+$$
$$x \mapsto \sum_{i=1}^{d} |\pi_i(x)|$$

where $x = \sum_{i=1}^{d} \pi_i(x) e_i$ is the unique decomposition of v. Since V has finite dimension, $\|\cdot\|_V$ and N are equivalent, and there exists $\alpha > 0$ such that $\frac{1}{\alpha}\|x\| \leq N(x) \leq \alpha\|x\|$ for all $x \in V$.

Let (x_n) be a sequence of elements of V, and assume (x_n) converges to $x \in X$, i.e. $\lim_n \|x - x_n\| = 0$. For all n, let the unique decomposition of x_n in the basis (e_i) be given by

$$x_n = \sum_{i=1}^{d} x_{n,i} e_i$$

where $x_{n,i} = \pi_i(x_n)$. Then we have for all i, $(x_{n,i})_n$ is a Cauchy sequence since

$$\begin{aligned} |x_{n,i} - x_{m,i}| &\leq N(x_n - x_m) \\ &\leq \alpha \|x_n - x_m\| && \text{by equivalence of the norms on } V \end{aligned}$$

Since $\mathbb{F}$ is complete, $(x_{n,i})_n$ converges. Let x_i be its limit. Then we have

$$\begin{aligned} \|\sum_{i=1}^d x_{n,i}e_i - \sum_{i=1}^d x_i e_i\| &\leq \alpha N\left(\sum_{i=1}^d (x_{n,i} - x_i)e_i\right) \\ &= \alpha \sum_{i=1}^d |x_{n,i} - x_i| \end{aligned}$$

which converges to 0 as $n \to \infty$, therefore

$$\sum_{i=1}^d x_i e_i = \lim_{n\to\infty} \sum_{i=1}^d x_{n,i} e_i = \lim_{n\to\infty} x_n = x$$

therefore x is a linear combination of the basis vectors (e_i), i.e. $x \in V$.

(b) Let $B = \{x \in X : \|x\|_X \leq 1\}$. Show that if B is compact, then X has finite dimension.

proof We show the contrapositive. Suppose that X has infinite dimension. We construct a sequence (x_n) of elements of B such that for all n, $d(x_n, \operatorname{span}(x_1, \ldots, x_{n-1}) > 1/2$.

- Let $x_1 = 0$
- Suppose the term x_n constructed. Then consider the subspace $V_n = \operatorname{span}(x_1, \ldots, x_n)$. We have V is closed by part (a), and it is a proper subspace since X has infinite dimension. Therefore by (10.6), there exists $x \in X$ such that

  $$d(x, V_n) > \frac{1}{2}\|x\|$$

  in particular x must be non-zero (since $0 \in V_n$). Let $x_{n+1} = \frac{x}{\|x\|}$. Then we have $x_{n+1} \in B$, and

  $$\begin{aligned} d(x_{n+1}, V_n) &= \inf_{v\in V_n} \|x_{n+1} - v\| \\ &= \inf_{v\in V_n} \|\frac{x}{\|x\|} - v\| \\ &= \inf_{v'\in V_n} \frac{1}{\|x\|}\|x - v'\| \\ &= \frac{1}{\|x\|} d(x, V_n) \\ &> \frac{1}{2} \end{aligned}$$

this completes the construction of (x_n). Then any subsequence of (x_n) does not converge, since for all $n > m$, $x_m \in V_{n-1}$, thus

$$\|x_n - x_m\| \geq d(x_n, V_{n-1}) > 1/2$$

therefore (x_n) is a sequence of elements of B that has no converging subsequence, and it follows that B is not compact.

(10.8) Let X, Y be Banach spaces. Let B be the closed unit ball in X

$$B = \{x \in X : \|x\|_X \leq 1\}$$

A bounded linear operator $L \in B(X, Y)$ is said to be compact if $\text{cl}(L(B))$ is a compact subset of Y.

Note : To show that L is compact, it suffices to show that $L(B)$ is totally bounded. Indeed, if $L(B)$ is totally bounded, then so is $\text{cl}(L(B))$: for all $\epsilon > 0$, we can cover $L(B)$ with finitely many balls of radius $\epsilon/2$, i.e.

$$L(B) \subset \cup_{i=1}^{k} B(y_i, \epsilon/2)$$

then taking the closure

$$\begin{aligned}\text{cl}(L(B)) &\subset \text{cl}(\cup_{i=1}^{k} B(y_i, \epsilon/2)) \\ &= \cup_{i=1}^{k} \text{cl}(B(y_i, \epsilon/2)) \\ &= \cup_{i=1}^{k} B(y_i, \epsilon)\end{aligned}$$

and $\text{cl}(L(B))$ can be covered with finitely many balls of radius ϵ. Finally, $\text{cl}(L(B))$ is closed and totally bounded, hence compact (this is true in any complete metric space).

(a) Give an example of a bounded linear operator which is not compact

answer Consider $X = \ell^\infty$, and $L = id$. Then L is bounded linear, and by (10.8), since the space has infinite dimension, $\text{cl}(L(B)) = \text{cl}(B) = B$ is not compact.

(b) L is said to have finite rank if the dimension of its range is finite. Show that any bounded linear operator of finite rank is compact.

proof Let L be a bounded linear operator of finite rank. Then $L(X)$ is a finite dimensional space. First, $L(B)$ is bounded since for all $y \in L(B)$, there exists $x \in B$ such that $y = L(x)$, and

$$\|y\|_Y = \|L(x)\| \leq \|L\|\|x\|_X \leq \|L\|$$

thus $L(B)$ is bounded, and so is $\text{cl}(L(B))$. Next, we have $L(B) \subset L(X)$, thus $\text{cl}(L(B)) \subset \text{cl}(L(X)) = L(X)$ since $L(X)$ is closed as a finite dimensional subspace.

Therefore $\text{cl}(L(B))$ is a closed and bounded subset of the finite dimensional subspace $L(X)$, therefore it is compact.

(c) Give an example of a bounded linear operator which does not have finite rank, but is compact.

answer Consider the Banach space of bounded real-valued sequences, $X = \ell^\infty$, and the linear operator

$$\begin{aligned}L : \ell^\infty &\to \ell^\infty \\ (x_1, x_2, \dots) &\mapsto L(x) = \left(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots\right)\end{aligned}$$

we have

- L is bounded. In fact $\|L\| = 1$
- L has infinite rank

- L is compact: by the note, it suffices to show that $L(B)$ is totally bounded. Let $\epsilon > 0$, and let N be the smallest integer such that $1/N < \epsilon$. Then for all $n \geq N$ and for all $y \in L(B)$, there exists $x \in B$ such that $y = L(x)$, and

$$|y_n| = |\frac{1}{n}x_n| \leq \frac{1}{n}\|x\|_\infty < \epsilon$$

Now consider sequences of the form

$$x = (x_1, \dots x_{N-1}, 0, 0, \dots)$$

where for all $n \leq N-1$, $x_n \in \mathbb{Z}\epsilon \cap [-\frac{1}{n}, \frac{1}{n}]$. Let S_ϵ be the set of all such sequences. S_ϵ is finite since there are finitely many choices for each term x_n for all $n \leq N-1$.

Now we have $\{B(x,\epsilon), x \in S_\epsilon\}$ covers $L(B)$. Indeed, for all $y = L(x) \in L(B)$

  – for all $n \geq N$, $|y_n - 0| < \epsilon$
  – for all $n \leq N-1$, $|y_n| = \frac{1}{n}|x_n| \leq \frac{1}{n}$, thus there exists k_n such that $-\frac{1}{n} \leq k_n\epsilon \leq y_n < (k_n+1)\epsilon \leq \frac{1}{n}$, in particular, $|y_n - k_n\epsilon| < \epsilon$

let $x = (k_1\epsilon, k_2\epsilon, \dots, k_{N-1}\epsilon, 0, 0, \dots)$. Then $x \in S$, and $\|y - x\|_\infty < \epsilon$, i.e. $y \in B(x,\epsilon)$, which proves the claim that $\{B(x,\epsilon), x \in S_\epsilon\}$ covers $L(B)$.

(d) Show that if $L_n \in B(X,Y)$ are compact, if $L \in B(X,Y)$, and $L_n \to L$ in the sense that $\|L_n - L\|_{B(X,Y)} \to 0$, then L is compact

proof By the note, it suffices to show $L(B)$ is totally bounded. Let $\epsilon > 0$. Since $L_n \to L$, there exists N such that for all $n \geq N$, $\|L_n - L\|_{B(X,Y)} \leq \epsilon/2$. In particular, we have

$$\|L_N - L\|_{B(X,Y)} \leq \epsilon/2 \tag{3}$$

Now consider L_N. Since $\mathrm{cl}(L_N(B))$ is compact, it is totally bounded, thus it can be covered by finitely many balls of radius $\epsilon/2$. Thus there exist $y_1, \dots, y_k$ such that

$$L_N(B) \subset \mathrm{cl}(L_N(B)) \subset \cup_{i=1}^k B(y_i, \epsilon/2)$$

i.e. for all $x \in B$, there exists $i \in \{1, \dots, k\}$ such that

$$\|L_N(x) - y_i\|_Y < \epsilon/2$$

and we have for such i

$$\begin{aligned}
\|L(x) - y_i\|_Y &\leq \|L(x) - L_N(x)\|_Y + \|L_N(x) - y_i\|_Y && \text{by the triangle inequality} \\
&\leq \|L - L_N\|_{B(X,Y)}\|x\|_X + \epsilon/2 \\
&< \epsilon/2 + \epsilon/2 && \text{by (3) and the fact that } x \in B
\end{aligned}$$

therefore $L(B)$ is covered by $B(y_i, \epsilon)$. This proves $L(B)$ is totally bounded, and thus L is compact.

(e) Show that if X, Y, Z are Banach spaces and $S : X \to Y$ and $T : Y \to Z$ are bounded linear operators, and if either S or T is compact, then $S \circ T$ is compact.

proof

- suppose T is compact. Then $\mathrm{cl}(T(B_X))$ is compact, and since S is continuous, $S(\mathrm{cl}(T(B_X)))$ is compact. As a consequence, it is also totally bounded. Now we have

$$S(T(B_X)) \subset S(\mathrm{cl}(T(B_X)))$$

thus $S(T(B_X))$ is a subset of a totally bounded set, and is totally bounded. By the note, $S \circ T$ is compact.

- suppose S is compact. We have for all $y \in T(B_X)$, there exists $x \in B_X$ such that $y = T(x)$, and $\|y\|_Y \leq \|T\|\|x\|_X \leq \|T\|$, thus $\frac{1}{\|T\|}y \in B_Y$, i.e.

$$T(B_X) \subset \frac{1}{\|T\|}B_Y$$

therefore $S(T(B_X)) \subset \frac{1}{\|T\|}S(B_Y)$ by linearity of S, and taking the closure

$$\mathrm{cl}(S(T(B_X))) \subset \mathrm{cl}\left(\frac{1}{\|T\|}S(B_Y)\right) = \frac{1}{\|T\|}\,\mathrm{cl}(S(B_Y))$$

which is compact since $\mathrm{cl}(S(B_Y))$ is compact (S is compact), and $z \mapsto \frac{1}{\|T\|}z$ is continuous. This proves that $S \circ T$ is compact.

(f) Show that if $K \in B(X, Y)$ is compact and if T has an inverse in $B(Y, X)$, then $T + K$ has finite-dimensional null-space.

proof Let $N = \ker(T + K)$ be the nullspace of $T + K$. N is a subspace. We have for all $x \in N$

$$(T + K)(x) = 0$$

thus applying T^{-1}

$$x + T^{-1} \circ K(x) = 0$$

now $(T^{-1} \circ K)_{|N}$ is a compact linear operator by (e), thus $\mathrm{cl}(T^{-1} \circ K(B_N))$ is compact (where B_N is the unit ball in N). But

$$(T^{-1} \circ K)_{|N} = -id_N$$

thus

$$\mathrm{cl}(T^{-1} \circ K(B_N)) = \mathrm{cl}(-id(B_N)) = B_N$$

which proves that B_N is compact, and by (7.b), N is finite dimensional.

(g) Show that under the same assumptions, the range $\mathcal{R}$ of $T + K$ is closed.

- Let (y_n) be a sequence in $\mathcal{R}$, and assume that y_n converges to $y \in Y$. For all n, there exists $x_n \in X$ such that $(T + K)(x_n) = y_n$, and composing with T^{-1}, we have

$$x_n + T^{-1} \circ K(x_n) = T^{-1}(y_n)$$

we observe that if (x_n) is bounded by M, then $x_n/M \in B_X$ for all n, and since $T^{-1} \circ K$ is compact, it follows that $T^{-1} \circ K(x_n/M) \in \mathrm{cl}(T^{-1} \circ K(B_X))$, which is compact. Therefore it has a converging sequence

$$(T^{-1} \circ K(x_{n_k}/M))_k$$

and since $T^{-1}(y_{n_k})$ is also converging (T^{-1} is continuous and y_n converges), we have

$$x_{n_k} = -T^{-1} \circ K(x_{n_k}) + T^{-1}(y_{n_k})$$

converges. Let x be its limit. Then by continuity of $T + K$, we have

$$(T + K)(x) = \lim(T + K)(x_{n_k}) = \lim y_{n_k} = y$$

which proves that $y \in \mathcal{R}$.

Now we show that it is possible to choose the elements x_n such that (x_n) is bounded. The idea is to choose x_n in a complement of the nullspace N.

- Existence of a complement of N: by (f), we have $N = \ker(T+K)$ has finite dimension. We first use the Hahn-Banach theorem to obtain a projection onto N. Let $(e_1, \dots, e_d)$ be a basis for N. For all i, let

$$\begin{aligned} \ell_i : N &\to \mathbb{R} \\ x_1 e_1 + \dots x_d e_d &\mapsto x_i \end{aligned}$$

we have ℓ_i is linear bounded, thus by the Hahn-Banach theorem, there exists a bounded linear functional $\bar{\ell}_i$ that extends ℓ_i to X. Now consider

$$\begin{aligned} \pi : X &\to N \\ x \mapsto \pi(x) &= \sum_{i=1}^{d} \bar{\ell}_i(x) e_i \end{aligned}$$

then π is linear, has range N, $\pi_{|N} = id_N$ since $\pi(e_i) = e_i$ for all $i \in \{1, \dots, d\}$, and $\ker(\pi) = \cap_{i=1}^{d} \ker(\bar{\ell}_i)$

$$X = N \oplus \ker(\pi)$$

since for all $x \in X$, we can decompose

$$x = \pi(x) + x - \pi(x)$$

where $x - \pi(x) \in \ker(\pi)$ since $\pi(x - \pi(x)) = \pi(x) - \pi \circ \pi(x) = 0$, and the decomposition is unique since $x \in N \cap \ker(\pi) \Rightarrow x = \pi(x) = 0$.

- It follows that the restriction of $T + K$ to $\ker(\pi)$ is injective. Let $(T + K)^\dagger$ be the inverse of the function

$$\begin{aligned} \ker(\pi) &\to \mathcal{R} \\ x &\mapsto (T+K)(x) \end{aligned}$$

we have $(T + K)^\dagger$ is a bounded linear operator.

- Now when constructing the sequence (x_n), we choose $x_n = (T+K)^\dagger(y_n)$. Then we have $x_n \in \ker(\pi)$ and $(T+K)(x_n) = y_n$. Then x_n is bounded. Indeed, we have (y_n) is bounded (since it is a converging subsequence) and $(T + K)^\dagger$ is a bounded linear operator.

(h) Show that $\mathcal{R}$ has finite codimension, that is, there exist finitely many elements $x_i \in X$ such that

$$X = \mathcal{R} \oplus \text{span}(x_1, \dots x_n)$$