MATH 202A - Problem Set 6

Walid Krichene (23265217)

October 3, 2012

(6.1) Let X, Y be compact metric spaces. Then for any $f \in C(X \times Y, \mathbb{C})$ and any $\epsilon > 0$, there exists $N \in \mathbb{N}$ and functions $g_i \in C(X, \mathbb{C})$ and $h_i \in C(Y, \mathbb{C})$ such that $\phi(x, y) = \sum_{i=1}^N g_i(x)h_i(y)$ satisfies $|f(x, y) - \phi(x, y)| < \epsilon$ for all $(x, y) \in X \times Y$

proof Consider the set of functions $\mathcal{A} = \{\phi \in C(X \times Y, \mathbb{C}) | \exists N \in \mathbb{N}, g_i \in C(X, \mathbb{C}), h_i \in C(Y, \mathbb{C}) : \forall (x, y) \in \phi(x, y) = \sum_{i=1}^N g_i(x)h_i(x)\}$. Then we have

- $\mathcal{A}$ is an algebra since it is closed under
  - scalar multiplication: for all $\lambda \in \mathbb{C}$, $\lambda g_1 \in C([0, 1], \mathbb{C})$ if $g_1 \in C([0, 1], \mathbb{C})$, thus $\lambda\phi(x, y) \in \mathcal{A}$
  - addition: $\sum_{i=1}^N g_i(x)h_i(y) + \sum_{j=N+1}^{N+N'} g_j(x)h_j(y) = \sum_{i=1}^{N+N'} g_i(x)h_i(y)$
  - pointwise product:

$$\left(\sum_{i=1}^N g_i(x)h_i(y)\right)\left(\sum_{j=1}^{N'} q_j(x)r_j(y)\right) = \sum_{(i,j)\in\{1,\dots,N\}\times\{1,\dots,N'\}} g_i(x)q_j(x)h_i(y)r_j(y)$$
$$= \sum_{(i,j)\in\{1,\dots,N\}\times\{1,\dots,N'\}} s_{i,j}(x)t_{i,j}(y)$$

    where for all $x \in X$, $s_{i,j}(x) = g_i(x)q_j(x)$ ($s_{i,j} \in C(X, \mathbb{C})$), and for all $y \in Y$, $t_{i,j}(y) = h_i(y)r_j(y)$ ($t_{i,j} \in C(Y, \mathbb{C})$)

- $\mathcal{A}$ contains constants, since a constant function can be written $\phi(x, y) = g_1(x)h_1(y)$ where g_1, h_1 are both constants (thus $g_1 \in C(X, \mathbb{C})$ and $h_1 \in C(Y, \mathbb{C})$)

- $\mathcal{A}$ is closed under taking the conjugate: $\bar{\phi}(x, y) = \sum_{i=1}^N \bar{g}_i(x)\bar{h}_i(x)$, and for all i, $\bar{g}_i \in C(X, \mathbb{C})$ and $\bar{h}_i \in C(X, \mathbb{C})$

- $\mathcal{A}$ separates points: let $p_1, p_2 \in X \times Y$, $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$, and assume that $p_1 \neq p_2$, i.e. either $x_1 \neq x_2$, or $y_1 \neq y_2$, in other words, $d(x_1, x_2) + d(y_1, y_2) > 0$. Then consider the function

$$\phi(x, y) = d(x_1, x) + d(y_1, y)$$

  We have $\phi \in \mathcal{A}$ since $x \mapsto d(x_1, x)$ is continuous, and so is $y \mapsto d(y, y_1)$, $\phi(x_1, y_1) = 0$, and $\phi(x_2, y_2) = d(x_1, x_2) + d(y_1, y_2) > 0$. Therefore ϕ separates p_1 and p_2.

- $X \times Y$ is compact: let $((x_n, y_n))_n$ be a sequence of elements of $X \times Y$. Since X is compact, $(x_n)_n$ has a converging subsequence $(x_{\phi(n)})_n$. Let x be its limit. Since Y is compact, $(y_{\phi(n)})_n$ has a converging subsequence $(y_{\phi(\psi(n))})_n$. Let y be its limit. Now since $(x_{\phi(\psi(n))})_n$ is a subsequence of the converging sequence $x_{\phi(n)}$, it also converges to the same limit x. Finally, $((x_{\phi(\psi(n))}, y_{\phi(\psi(n))}))_n$ converges to (x, y), and is a subsequence of $((x_, y_n))_n$.

Therefore by the Stone Weirstrass theorem, $\mathcal{A}$ is dense in $C = C(X \times Y, \mathbb{C})$ equipped with the metric $d(\phi_1, \phi_2) = \sup_{(x,y)\in X\times Y} |\phi_1(x, y) - \phi_1(x, y)|$, thus for every $f \in C$, and for all $\epsilon > 0$ there exists $\phi \in \mathcal{A}$ such that $\sup_{(x,y)\in X\times Y} |f(x, y) - \phi(x, y)| \leq \epsilon$, which proves the desired result.

(6.2) Let $X = C([0,1],\mathbb{R})$ with the metric $d(f,g) = \int_0^1 |f(x) - g(x)|dx$. Then X is not complete.

proof Consider the sequence of functions $f_n \in X$ defined by

$$f_n(x) = \begin{cases} 0 & \text{if } x \in [0, 1/2 - 1/n] \\ 1/2 + n(x - 1/2) & \text{if } x \in [1/2 - 1/n, 1/2 + 1/n] \\ 1 & \text{if } x \in [1/2 + 1/n, 1] \end{cases}$$

The sequence $(f_n)_n$ is Cauchy since for all $n > m$,

$$\begin{aligned} d(f_n, f_m) &= \int_{x=1/2-1/m}^{1/2+1/m} |f_n(x) - f_m(x)|dx \\ &\leq \int_{x=1/2-1/m}^{1/2+1/m} |f_n(x)| + |f_m(x)|dx \\ &\leq \int_{x=1/2-1/m}^{1/2+1/m} 2dx \\ &\leq 4/m \end{aligned}$$

and converges to 0 when $m \to \infty$.

However, $(f_n)_n$ does not converge in $C([0,1],\mathbb{R})$. Indeed, let $f \in C([0,1],\mathbb{R})$. We show that $d(f_n, f)$ is bounded below by a positive number, i.e. (f_n) cannot converge to f. Let $\epsilon = 1/3$. By continuity of f at $x = 1/2$, there exists $\delta > 0$ such that if $|x - 1/2| \leq \delta$, then $|f(x) - f(1/2)| \leq \epsilon$. In particular, the points $x_1 = 1/2 - \delta$ and $x_2 = 1/2 + \delta$ satisfy: $|f(x_1) - f(x_2)| \leq |f(x_1) - f(1/2)| + |f(1/2) - f(x_2)| \leq 2\epsilon < 1$. Therefore we must have either $f(x_1) > 0$ or $f(x_2) < 1$ (otherwise $|f(x_1) - f(x_2)| \geq 1$). Now we consider the two cases separately

- first case: $f(x_1) > 0$, $x_1 < 1/2$. Take $\epsilon = f(x_1)/2$ By continuity of f at x_1, there exists $\eta \in (0, 1/2 - x_1)$ such that $\forall x \in [x_1 - \eta, x_1 + \eta]$, $f(x) \geq f(x_1) - \epsilon = f(x_1)/2$. Now taking $N \in \mathbb{N}$ such that $1/2 - 1/N > x_1 + \eta$ (this is possible since $\eta < 1/2 - x_1$, thus $1/2 - x_1 - \eta > 0$ and it suffices to take $N > 1/(1/2 - x_1 - \eta)$), we have for all $n \geq N$, and for all $x \in [x_1 - \eta, x_1 + \eta]$, $x_1 + \eta < 1/2 - 1/N \leq 1/2 - 1/n$, thus $f_n(x) = 0$ (by definition of f_n). Therefore

$$\begin{aligned} d(f_n, f) &= \int_{x=0}^{1} |f(x) - f_n(x)|dx \\ &\geq \int_{x=x_1-\eta}^{x_1+\eta} |f(x) - f_n(x)|dx \\ &\geq \int_{x=x_1-\eta}^{x_1+\eta} |f(x)|dx \\ &\geq \int_{x=x_1-\eta}^{x_1+\eta} \frac{f(x_1)}{2}dx \\ &= 2\eta\frac{f(x_1)}{2} \\ &= \eta f(x_1) > 0 \end{aligned}$$

- second case: $f(x_2) < 1$, $x_2 > 1/2$. A similar argument shows that $d(f, f_n)$ is bounded below by a positive number $\eta(1 - f(x_2)) > 0$.

This shows that $(f_n)_n$ cannot converge to any continuous function $f \in C([0,1],\mathbb{R})$.

(6.3) Let $f : [0,1] \to \mathbb{C}$ be continuous. Then if $\int_0^1 f(x)x^n dx = 0$ for all $n \in \mathbb{N}$, then f is identically 0.

proof Consider the sub-algebra $\mathcal{P} \subset C([0,1],\mathbb{C})$ of complex valued polynomial functions on $[0,1]$. We have $\mathcal{P}$ contains the constants, is closed under taking the conjugate, and separates points. Thus by the Stone-Weierstrass theorem, $\mathcal{P}$ is dense in $C([0,1],\mathbb{C})$ equipped with the metric $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$.

First, we show that for all $p \in \mathcal{P}$, $\int_0^1 f(x)p(x)dx = 0$. Let $p \in \mathcal{P}$. Then there exist $n \in \mathbb{N}$ and $a_0, \dots, a_n \in \mathbb{C}$ such that $p(x) = \sum_{i=0}^n a_n x^n$, and we have by linearity of the integral

$$\begin{aligned}\int_0^1 f(x)p(x)dx &\int_0^1 f(x) \sum_{i=0}^n a_n x^n dx \\ &\sum_{i=0}^n a_n \int_0^1 f(x)x^n dx \\ &= 0\end{aligned}$$

by assumption on f.

Now let $\epsilon > 0$. We want to show that $\int_0^1 |f(x)|^2 dx \leq \epsilon$. Since f is continuous on the compact $[0,1]$, it is bounded, i.e. there exists M such that $|f(x)| \leq M$ for all $x \in [0,1]$. Since $\mathcal{A}$ is dense in $C([0,1],\mathbb{C})$, then there exists a polynomial function $p \in \mathcal{P}$ such that $\sup_{x\in[0,1]} |f(x) - p(x)| \leq \epsilon/M$. Then we have

$$\begin{aligned}\int_0^1 |f(x)|^2 dx &= \int_0^1 f(x)\overline{f(x)}dx \\ &= \int_0^1 f(x)\overline{(f(x) - p(x) + p(x))}dx \\ &= \int_0^1 f(x)\overline{p(x)} + \int_0^1 f(x)\overline{(f(x) - p(x))}dx \\ &= \int_0^1 f(x)\overline{(f(x) - p(x))}dx\end{aligned}$$

by the previous result, since $\bar{p} \in \mathcal{P}$. Then

$$\begin{aligned}|\int_0^1 f(x)\overline{(f(x) - p(x))}dx| &\leq \int_0^1 |f(x)||f(x) - p(x)|dx \\ &\leq \int_0^1 M\epsilon/M dx \\ &= \epsilon\end{aligned}$$

which gives the desired bound, $\int_0^1 |f(x)|^2 dx \leq \epsilon$. Finally, since this is true for all $\epsilon > 0$, then

$$\int_0^1 |f(x)|^2 dx = 0$$

this implies that f is identically zero. Indeed, assume by contradiction that there exists $x_0 \in [0,1]$ such that $|f(x_0)| > 0$. Then by continuity of f at x_0, there exists $\delta \in (0,1)$ such that for all $x \in [0,1] \cap [x_0 - \delta, x_0 + \delta]$, $|f(x) - f(x_0)| \leq |f(x_0)|/2$, thus by the triangle inequality $|f(x)| \geq |f(x_0)| - |f(x) - f(x_0)| \geq |f(x_0)| -$

$|f(x_0)|/2 = |f(x_0)|/2$. And we have

$$\begin{aligned}\int_0^1 |f(x)|^2 dx &\geq \int_{[0,1]\cap[x_0-\delta,x_0+\delta]} |f(x)|^2 dx \\ &\geq \int_{[0,1]\cap[x_0-\delta,x_0+\delta]} |f(x_0)|^2/4 dx \\ &\geq \delta |f(x_0)|^2/4 > 0\end{aligned}$$

which contradicts the fact that $\int_0^1 |f(x)|^2 dx = 0$ and completes the proof.

(6.4.a) Let $X = \{z \in \mathbb{C} : |z| = 1\}$, equipped with the metric which it inherits as a subset of $\mathbb{C}$. Let $\mathcal{A} \subset C(X, \mathbb{C})$ be the set of all functions of the form $\sum_{n=0}^N a_n z^n$ where N varies over all non-negative integers and each $a_n \in \mathbb{C}$. Note that $\mathcal{A}$ is an algebra of continuous functions, which separates points of X. But $\mathcal{A}$ is not a dense subset of $C(X, \mathbb{C})$ (with respect to the uniform metric).

proof We first observe that for all $g \in \mathcal{A}$, $\int_0^{2\pi} g(e^{i\theta}) e^{i\theta} d\theta = 0$. Let $g(z) = \sum_{n=0}^N a_n z^n$. Then by linearity of the integral

$$\begin{aligned}\int_0^{2\pi} g(e^{i\theta}) e^{i\theta} d\theta &= \sum_{n=0}^N a_n \int_0^{2\pi} (e^{i\theta})^n e^{i\theta} d\theta \\ &= \sum_{n=0}^N a_n \int_0^{2\pi} e^{i(n+1)\theta} d\theta \\ &= \sum_{n=0}^N a_n \left[\frac{1}{i(n+1)} e^{i(n+1)\theta}\right]_0^1 \\ &= 0\end{aligned}$$

Therefore for any function f in the closure of $\mathcal{A}$ (with respect to the uniform metric), we also have $\int_0^1 f(e^{i\theta}) e^{i\theta} d\theta = 0$. Indeed, let $f \in \mathrm{cl}(\mathcal{A})$, and let (f_n) be a sequence of functions in $\mathcal{A}$ that converges uniformly to f. Then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, for all $z \in X$, $|f(z) - f_n(z)| \leq \epsilon$. Therefore

$$\begin{aligned}|\int_0^1 f(e^{i\theta}) e^{i\theta} d\theta| &= |\int_0^1 f(e^{i\theta}) e^{i\theta} d\theta - \int_0^1 f_n(e^{i\theta}) e^{i\theta} d\theta| \\ &= |\int_0^1 (f(e^{i\theta}) - f_n(e^{i\theta})) e^{i\theta} d\theta| \\ &\leq \int_0^1 |f(e^{i\theta}) - f_n(e^{i\theta})||e^{i\theta}| d\theta \\ &\leq \epsilon\end{aligned}$$

Since this is true for all $\epsilon > 0$, we have $\int_0^1 f(e^{i\theta}) e^{i\theta} d\theta = 0$. (also follows from the fact that if (f_n) converges uniformly to f, and h is bounded, then $(\int f_n h)$ converges to $\int fh$).

But this is not the case for the continuous function

$$\begin{aligned}f : X &\to \mathbb{C} \\ z &\mapsto 1/z\end{aligned}$$

since $\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} d\theta = \int_0^{2\pi} 1 d\theta = 2\pi$. This proves that not all continuous functions are in the closure of $\mathcal{A}$.

(6.4.b) Let $\mathcal{B}$ be the set of all functions of the form $\{\sum_{n=-N}^{N} a_n z^n, N \in \mathbb{N}, a_n \in \mathbb{C}\}$. Then $cl(\mathcal{B}) = C(X, \mathbb{C})$.

proof We have

- $\mathcal{B}$ is an algebra, since it is closed under
  - scalar multiplication (clear)
  - addition (clear)
  - pointwise product
  $$\left(\sum_{n=-N}^{N} a_n z^n\right)\left(\sum_{-M}^{M} b_m z^m\right) = \sum_{k=-(M+N)}^{M+N} c_k z^k$$
  where $c_k = \sum_{n+m=k} a_n b_m$.
- $\mathcal{B}$ is closed under taking the conjugate: $\forall z \in X$, $\bar{z} = 1/z \in X$, thus $\overline{\sum_{n=-N}^{N} a_n z^n} = \sum_{n=-N}^{N} \bar{a}_n z^{-n} = \sum_{m=-N}^{N} c_m z^m$ where $c_m = \bar{a}_{-m}$.
- $\mathcal{B}$ contains constants (taking $N = 0$).
- $\mathcal{B}$ separates points: for any $z_1, z_2 \in X$, if $z_1 \neq z_2$, then the identity separates z_1 and z_2 and is a member of $\mathcal{B}$.

Therefore by the Stone-Weierstrass theorem, $\mathcal{B}$ is dense in the space of $C(X, \mathbb{C})$.

(6.5) Let p be a prime number, and consider the function $|\cdot|_p$ that maps any $r \in \mathbb{Q}$ to the $|r|_p = p^{-k}$ where $k \in \mathbb{Z}$ is the unique integer such that there exist $m \in \mathbb{Z}$, $n \in \mathbb{Z}^*$ such that $r = mp^k/n$, and p does not divide m or n. (i.e. m and n are coprime with p).

Then for all $r \in \mathbb{Q}$, and for all $s \in \mathbb{Q}$

- $|r|_p \geq 0$ with equality iff $r = 0$
- $|-r|_p = |r|_p$
- $|r+s|_p \leq \max\{|r|_p, |s|_p\}$ (i.e. $|\cdot|_p$ satisfies the ultrametric inequality)

proof

- By definition of $|\cdot|_p$, for all $r \neq 0$, $|r|_p = p^{-k} > 0$ for all $r \neq 0$, and $|0|_p = 0$. Therefore $|r|_p \geq 0$ with equality if and only if $r = 0$.
- Let $r \in \mathbb{Q}$, and let $r = p^k a/b$ a decomposition such that both p does not divide a or b. Then $|r|_p = p^{-k}$, and since $-r = p^k(-a)/b$ and p does not divide $-a$ or b, $|-r|_p = p^{-k}$, which proves that $|-r|_p = |r|_p$.
- Ultrametric inequality: Let $r, s \in \mathbb{Q}$, and let $r = p^{k_r} a_r/b_r$ and $s = p^{k_s} a_s/b_s$ be two decompositions (satisfying the required properties). Consider two cases: if $k_r = k_s$, then
  $$\begin{aligned} r + s &= p^{k_r}\left(\frac{a_r}{b_r} + \frac{a_s}{b_s}\right) \\ &= p^{k_r}\frac{a_r b_s + a_s b_r}{b_r b_s} \end{aligned}$$
  now let $a_r b_s + a_s b_r = p^k c$, where $k \geq 0$ and p does not divide c. Then
  $$r + s = p^{k_r + k}\frac{c}{b_r b_s}$$

where p does not divide c or $b_r b_s$ (p does not divide b_s or b_r so does not divide their product). Thus

$$|r+s|_p = p^{-k_r-k} \leq p^{-k_r} = \max(|r|_p, |s|_p)$$

If $k_r \neq k_s$, assume w.l.o.g. that $k_s > k_r$. Then

$$\begin{aligned} r+s &= p^{k_r}\left(\frac{a_r}{b_r} + p^{k_s-k_r}\frac{a_s}{b_s}\right) \\ &= p^{k_r}\frac{a_r b_s + p^{k_s-k_r}a_s b_r}{b_r b_s} \end{aligned}$$

p does not divide $b_r b_s$, and does not divide $a_r b_s + p^{k_s-k_r}a_s b_r$ (since $a_r b_s + p^{k_s-k_r}a_s b_r \equiv a_r b_s \mod (p)$ since $k_s - k_r > 0$, and p does not divide $a_r b_s$), thus

$$|r+s|_p = p^{-k_r} = \max(p^{-k_r}, p^{-k_s}) = \max(|r|_p, |s|_p)$$

(6.6) The multiplication,

$$\begin{aligned} m : \mathbb{Q} \times \mathbb{Q} &\to \mathbb{Q}_p \\ (a,b) &\mapsto ab \end{aligned}$$

extends to a continuous function $\bar{m} : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{Q}_p$

proof First, we show the following lemma:

Lemma 1 *Let (X,d) and (Y,ρ) be two metric spaces, and let $E \subseteq X$ be any dense subset of X. Let $f : E \to Y$ be a function that maps any Cauchy sequence of E that converges in X to a converging sequence of Y. Then f can be extended to a continuous function $\bar{f} : X \to Y$.*

proof of lemma For every point $x \in X$, define $\bar{f}(x)$ as the limit $\lim f(x_n)$ where (x_n) is any sequence of elements of E that converges to x (since E is dense in X, there exists such a sequence (x_n), and any such sequence is Cauchy in E. By assumption on f, $(f(x_n))_n$ is a converging sequence, and $\lim f(x_n)$ exists). To show that $\bar{f}$ is well defined, we need to verify that $\lim f(x_n)$ is independent of the choice of (x_n).

Let (x_n) and (x'_n) two Cauchy sequences of E that converge both to $x \in X$. Let $y = \lim f(x_n)$ and $y' = \lim f(x'_n)$. Then we want to show that $y = y'$. The sequence $(z_n) = (x_0, x'_0, x_1, x'_1, \dots)$ is a Cauchy sequence that converges to x, thus $(f(z_n))$ converges. Since $(f(x_n))$ and $(f(y_n))$ are both subsequences of $(f(z_n))$, they have the same limit, i.e. $y = y'$.

Note that this implies in particular that $\bar{f}$ agrees with f on E: for all $x \in E$, simply take the constant Cauchy sequence $x_n = x$ for all n. Then $\bar{f}(x) = \lim f(x_n) = f(x)$)

Continuity:

- $\bar{f}$ is continuous on X. By contradiction, assume that there exists $x \in X$ such that $\bar{f}$ is not continuous at x, i.e. there exists $\epsilon > 0$ such that for all $\delta > 0$, there exists x_δ such that $d(x, x_\delta) \leq \delta$ and $\rho(\bar{f}(x) - \bar{f}(x_\delta)) > \epsilon$. Then construct a sequence (x_n) as follows: for all $n \in \mathbb{N}$, choose $x_n \in X$ such that $d(x_n, x) \leq 1/(n+1)$ and $\rho(\bar{f}(x_n), \bar{f}(x)) > \epsilon$. Now since $x_n \in X$ and E is dense in X, there exists a sequence of elements $(a_k^{(n)})_k \in E$ such that $x_n = \lim_{k\to\infty} a_k^{(n)}$. And by definition of $\bar{f}$, $\bar{f}(x_n) = \lim_{k\to\infty} f(a_k^{(n)})$. Therefore there exists K such that for all $k \geq K$,

$$\begin{aligned} d(a_k^{(n)}, x_n) &\leq 1/(n+1) \\ \rho(\bar{f}(x_n), \bar{f}(a_k)) &\leq \epsilon/2 \end{aligned}$$

  Let $x'_n = a_K^{(n)}$. We then have using the triangle inequlity

$$\begin{aligned} d(x, x'_n) &\leq d(x, x_n) + d(x_n, x'_n) \leq 2/(n+1) \\ \rho(\bar{f}(x), f(x_n)) &\geq \rho(\bar{f}(x), \bar{f}(x_n)) - \rho(\bar{f}(x_n), \bar{f}(x'_n)) > \epsilon - \epsilon/2 = \epsilon/2 \end{aligned}$$

Therefore (x'_n) is a Cauchy sequence in E that converges to x, however $(f(x'_n))$ does not converge to $\bar{f}(x)$. This contradicts the definition of $\bar{f}$. Therefore $\bar{f}$ is continuous.

- Note: the definition of $\bar{f}$ and the fact it agrees with f on E imply in particular that f is continuous on E: Let $x \in E$, and let $(x_n)_n$ be any sequence of E that converges to x. Then $f(x) = \bar{f}(x) = \lim f(x_n)$ (f and $\bar{f}$ agree on E), thus f is continuous on E.

***proof of* (6.6)** We have $\mathbb{Q} \times \mathbb{Q}$ is dense in $\mathbb{Q}_p \times \mathbb{Q}_p$ (equipped with the metric $d((a,b),(a',b')) = |a-a'|_p + |b-b'|_p$). Thus by the previous lemma, it suffices to show that for any Cauchy sequence $((a_n, b_n))$ in $\mathbb{Q} \times \mathbb{Q}$, $(a_n b_n)_n$ converges in $\mathbb{Q}_p$. First, (a_n) is a Cauchy sequence in $(\mathbb{Q}, |\cdot|_p)$ since $|a_n, a_m|_p \le d((a_n, b_n), (a_m, b_m))$, and so is (b_n), thus both sequences are bounded, i.e. there exist $A, B \in \mathbb{Q}$ such that $|a_n|_p \le A$ and $|b_n|_p \le B$ for all n. We have

$$\begin{aligned} |a_n b_n - a_m b_m|_p &\le |a_n b_n - a_n b_m|_p + |a_n b_m - a_m b_m|_p \\ &\le |a_n|_p |b_n - b_m|_p + |b_m|_p |a_n - a_m|_p \\ &\le A|b_n - b_m|_p + B|a_n - a_m|_p \end{aligned}$$

and since $|b_n - b_m|_p$ and $|a_n - a_m|$ both converge to zero when $m, n \to \infty$, so does $|a_n b_n - a_m b_m|_p$, and $(a_n, b_n)_n$ is Cauchy in $\mathbb{Q}_p$ which is complete, thus converges. This completes the proof.

(6.7) Let $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$, and $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$. Then the function

$$\begin{aligned} f : \mathbb{Q}^* &\to \mathbb{Q}_p^* \\ r &\mapsto 1/r \end{aligned}$$

extends to a continuous function $\bar{f} : \mathbb{Q}_p^* \to \mathbb{Q}_p^*$

proof We have $\mathbb{Q}^*$ is dense in $\mathbb{Q}_p^*$, thus by the previous lemma (applied to $f : \mathbb{Q}^* \to \mathbb{Q}_p$), it suffices to show that any Cauchy sequence (x_n) in $\mathbb{Q}^*$ that converges to $x \in \mathbb{Q}_p^*$, $(f(x_n))$ converges in $\mathbb{Q}_p^*$.

Let $x \in \mathbb{Q}_p^*$, and $(x_n) \in \mathbb{Q}^*$ a Cauchy sequence that converges to x. First, note that since $\lim x_n \neq 0$, $|x_n|_p$ is bounded below for n large enough. Indeed, let $\epsilon = |x|_p > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \ge N$, $|x_n - x|_p \le \epsilon/2$, thus

$$|x_n|_p \ge |x|_p - \epsilon/2 = \epsilon/2$$

Then we have for all $n, m \ge N$

$$\begin{aligned} |f(x_n) - f(x_m)|_p &= |1/x_n - 1/x_m|_p \\ &= |(x_m - x_n)/(x_n x_m)|_p \\ &\le |x_m - x_n|/|(x_n x_m)|_p \\ &\le |x_m - x_n|/(|(x_n|_p |x_m)|_p) \\ &\le |x_m - x_n|/\epsilon^2 \end{aligned}$$

where we use the fact that for all $r \in \mathbb{Q}^*$, $|1/r|_p = 1/|r|_p$ (since $|r|_p |1/r|_p = |r/r|_p = |1|_p = 1$).

Since $|x_m - x_n|_p$ converges to zero when $m, n \to \infty$, so does $|f(x_m) - f(x_n)|_p$, which proves that $(f(x_n))_n$ is Cauchy, thus it converges in $\mathbb{Q}_p$. To show that it converges in $\mathbb{Q}_p^*$, we simply observe that it cannot converge to 0: since (x_n) converges to $x \in \mathbb{Q}_p^*$, letting $\epsilon = |x|_p > 0$, there exists N such that for all $n \ge N$, $|x_n - x|_p \le \epsilon$, thus $|x_n|_p \le |x_n|_p + \epsilon = 2\epsilon$, thus for all $n \ge N$

$$|f(x_n)|_p = 1/|x_n|_p \ge 1/2\epsilon$$

therefore $(f(x_n))$ does not converge to 0, thus it converges in $\mathbb{Q}_p^*$. This completes the proof.

(6.8) The unit ball $B = \{t \in \mathbb{Q}_p | |t|_p \leq 1\}$ is totally bounded (thus compact, since it is also complete as a closed subset of the complete space $\mathbb{Q}_p$).

proof We show that for any $\epsilon > 0$, B has a finite cover with open balls of radii ϵ.

Let $\epsilon > 0$, and let $n \in \mathbb{N}$ be such that $p^{-n} < \epsilon$. Then we show that for any $x \in B \cap \mathbb{Q}$, there exists an integer $\bar{r} \in \{0, 1, \ldots, p^n - 1\}$ such that $|x - \bar{r}|_p \leq p^{-n}$.

First, note that for any integer v, $|v|_p \leq 1$ since v can be written uniquely as $v = p^{k_v} a$ with $k_v \geq 0$ and p does not divide a. Then $|v|_p = p^{-k_v} \leq 1$.

Let $x \in B \cap \mathbb{Q}$, and assume $x = p^k a/b$, such that p does not divide a or b. First, since $p^{-k} = |x|_p \leq 1$, then $k \geq 0$. Now since p does not divide b, p^n and b are coprime, thus by Bézout's theorem, there exist integers u, v such that

$$bu + p^n v = 1$$

Now let $r = p^k au$. We have

$$\begin{aligned}
|x - r|_p &= |p^k \frac{a}{b} - p^k au|_p \\
&= |p^k \frac{a}{b}|_p |1 - bu|_p \\
&\leq |x|_p |1 - bu|_p \\
&\leq |1 - bu|_p && \text{since } |x|_p \leq 1 \\
&\leq |p^n v|_p \\
&= |p^n|_p |v|_p \\
&\leq p^{-n} && \text{since } |v|_p \leq 1 \text{ since } v \text{ is integer}
\end{aligned}$$

Now let the Eucledian division of r by p^n be given by

$$r = qp^n + \bar{r},\ \ \bar{r} \in \{0, \ldots, p^n - 1\}$$

then

$$\begin{aligned}
|x - \bar{r}|_p &= |x - r + qp^n|_p \\
&\leq \max(|x - r|_p, |qp^n|_p) \\
&= \max(p^{-n}, p^{-n}|q|_p) \\
&\leq \max(p^{-n}, p^{-n}) && \text{since } q \text{ is integer} \\
&= p^{-n}
\end{aligned}$$

which shows that $|x - \bar{r}|_p \leq p^{-n}$.

Let $B_{\bar{r}} = \{x \in \mathbb{Q}_p | |x - \bar{r}|_p \leq p^{-n}\}$. We have $(B_{\bar{r}})_{\bar{r} \in \{0, \ldots, p^n - 1\}}$ forms a finite cover of $B \cap \mathbb{Q}$. It also forms a finite cover of B: let $x \in B$. By density of $\mathbb{Q}$ in $\mathbb{Q}_p$, there exists $x' \in \mathbb{Q}$ such that $|x - x'|_p \leq p^{-n}$. But $x' \in B_{\bar{r}}$ for some $\bar{r}$, and we have

$$\begin{aligned}
|x - \bar{r}|_p &= |x - x' + x' - \bar{r}|_p \\
&\leq \max(|x - x'|, |x' - \bar{r}|_p) \\
&\leq \max(p^{-n}, p^{-n}) \\
&= p^{-n}
\end{aligned}$$

thus $x \in B_{\bar{r}}$, therefore $(B_{\bar{r}})_{\bar{r} \in \{0, \ldots, p^n - 1\}}$ forms a finite cover of B. Finally, $(B(\epsilon, \bar{r}))_{\bar{r} \in \{0, \ldots, p^n - 1\}}$ forms a finite open cover of B, since $B_{\bar{r}} \subset B(\epsilon, \bar{r})$ $(p^{-n} < \epsilon)$.