MATH 202A - Problem Set 3

Walid Krichene

September 12, 2012

(3.1) Consider a metric space (X, d).

(a) Let (x_n) be a Cauchy sequence, and assume that a subsequence $(x_{\phi(n)})$ converges to $l \in X$. Then (x_n) also converges to l.

proof Let $\epsilon > 0$. Since x_n is a Cauchy sequence, then $\exists N_1 \in \mathbb{N}$ such that $\forall n, m \geq N_1$, $d(x_n, x_m) \leq \epsilon/2$. And since $(x_{\phi(n)})$ converges to l, then $\exists N_2 \in \mathbb{N}$ such that $\forall n > N_2$, $d(x_{\phi(n)}, l) \leq \epsilon/2$. Let $N = \max(N_1, N_2)$. Then we have $\forall n \geq N$, $d(x_n, l) \leq d(x_n, x_{\phi(n)}) + d(x_{\phi(n)}, l) \leq \epsilon/2 + \epsilon/2$ (since $\phi(n) \geq n$). Thus (x_n) converges to l.

(b) Let (x_n) be a Cauchy sequence. Then (x_n) is bounded.

proof Fix $\epsilon > 0$. We have $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N$, $d(x_n, x_m) \leq \epsilon$. In particular, we have $\forall n \geq N$, $d(x_n, x_N) \leq \epsilon$. Let $r = \max(\epsilon, d(x_0, x_N), \dots, d(x_{N-1}, x_N))$. Then we have $\forall n \in \mathbb{N}$, $d(x_n, x_N) \leq r$, thus (x_n) is bounded.

(3.2) Let (X, d) be a metric space, and $Y \subset X$. Let d' be the restriction of d to Y. Then if (Y, d') is complete, then Y is a closed subset of X.

proof Let (x_n) be a converging sequence of (X, d), such that $\forall n$, $x_n \in Y$, and let l be its limit. To show that Y is closed, it suffices to show that $l \in Y$ for any such sequence. First, (x_n) is a Cauchy sequence of (Y, d'): since (x_n) converges in (X, d), $\forall \epsilon > 0$, $\exists N$ such that $\forall n \geq N$, $d(x_n, l) \leq \epsilon/2$, thus $\forall n, m \geq N$, $d'(x_n, x_m) = d(x_n, x_m) \leq d(x_n, l) + d(l, x_m) \leq \epsilon$.

Since (Y, d') is complete, the Cauchy sequence (x_n) converges and its limit is in Y. Let $l' \in Y$ be the limit of (x_n) as a converging sequence of (Y, d'). Then l' is also a limit of (x_n) as a converging sequence of (X, d) since $\forall \epsilon > 0$, $\exists N$ such that $d'(x_n, l') \leq \epsilon$, thus $d(x_n, l') \leq \epsilon$. By uniqueness of the limit, we have $l = l'$, thus $l \in Y$.

(3.3) Let (X, d_X) and (Y, d_Y) be metric spaces, $f : X \to Y$ a continuous function, and $G = \{(x, y) \in X \times Y : y = f(x)\}$. Then G is closed subset of $(X \times Y, d)$ where d is the product metric $d(p_1, p_2) = \max(d_X(x_1, x_2), d_Y(y_1, y_2))$.

proof G is the inverse image of the closed set $\{0\}$ by the continuous function

$$\begin{aligned} h : (X, Y) &\to \mathbb{R} \\ (x, y) &\mapsto d(f(x), y) \end{aligned}$$

Indeed, $(x, y) \in G$ if and only if $f(x) = y$, if and only if $d(f(x), y) = 0$. h is continuous as the composition of the continuous functions h_1 and h_2 given by

$$X \times Y \xrightarrow{h_1} Y \times Y \xrightarrow{h_2} \mathbb{R}$$

where

$$\begin{aligned} h_1 : X \times Y &\to Y \times Y \\ (x, y) &\mapsto (f(x), y) \end{aligned}$$

$$h_2 : Y \times Y \to \mathbb{R}$$
$$(y, y') \mapsto d(y, y')$$

(3.4)

(a) Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $f : X \to Y$ be a homeomorphism. Then (x_n) converges in X if and only if $(f(x_n))$ converges in Y. Further, $x_n \to x$ if and only if $f(x_n) \to f(x)$.

proof Only if: let (x_n) be a converging sequence. Let x be its limit. Let $\epsilon > 0$. Since f is continuous, $\exists \eta > 0$ such that if $d_X(x_n, x) \leq \eta$, then $d_Y(f(x_n), f(x)) \leq \epsilon$. And since (x_n) converges to x, $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $d_X(x_n, x) \leq \eta$. Therefore $\forall n \geq N$, $d_Y(f(x_n), f(x)) \leq \epsilon$. This proves that $(f(x_n))$ converges, and that its limit is $f(x)$.

If: Assume $f(x_n)$ converges. Let y be its limit. Let $\epsilon > 0$. Since f^{-1} is continuous, there exists $\eta > 0$ such that if $d_Y(f(x_n), y) \leq \eta$, then $d_X(f^{-1}(f(x_n)), f^{-1}(y)) \leq \epsilon$. And since $(f(x_n))$ converges to y, $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $d_Y(f(x_n), y) \leq \eta$. Thus $\forall n \geq N$, $d_X(x_n, f^{-1}(y)) \leq \epsilon$, thus (x_n) converges, and its limit is $f^{-1}(y)$. Further, if $f(x_n) \to f(x)$, then $(x_n) \to f^{-1}(f(x)) = x$.

(b) $\mathbb{R}$ is not homeomorphic to $\mathbb{Q}$.

(*proof* since $\mathbb{R}$ is not countable, there are no injective maps from $\mathbb{R}$ to $\mathbb{Q}$. Therefore, there are no bijective maps from $\mathbb{R}$ to $\mathbb{Q}$.)

proof without using $\mathbb{R}$ uncountable: assume by contradiction that $f : \mathbb{Q} \to \mathbb{R}$ is a homeomorphism. Let (x_n) be a sequence of rationals that converge to an irrational x ((x_n) does not converge in $\mathbb{Q}$, but converges in $\mathbb{R}$). Then $f(x_n)$ is a converging sequence:

- (x_n) is a Cauchy sequence in $\mathbb{Q}$: let d be the usual metric on $\mathbb{R}$ and d' be its restriction on $\mathbb{Q}$. Then $\forall \epsilon > 0$, $\exists N$ such that $\forall n \geq N$ $d(x, x_n) \leq \epsilon/2$. Then $\forall m, n \geq N$, $d'(x_n, x_m) = d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) \leq \epsilon$.

- $f(x_n)$ is a Cauchy sequence in $\mathbb{R}$ as the image bu a continuous function of a Cauchy sequence. (since f is continuous, $\forall \epsilon > 0$, there exists $\eta > 0$ such that if $d(x, x') \leq \eta$ then $d(f(x), f(x')) \leq \epsilon$, then since (x_n) is Cauchy, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $d(x_n, x_m) \leq \eta$. Thus for all $m, n \geq N$, $d(f(x_n), f(x_m)) \leq \epsilon$.)

- $\mathbb{R}$ is complete. Thus $(f(x_n))$ converges.

Let y be the limit of $(f(x_n))$. But from (a), $(f(x_n))$ converges to y only if (x_n) converges to $f^{-1}(y)$. But (x_n) does not converge in $\mathbb{Q}$, contradiction.

(c) There is no bounded continuous injective mapping $f : \mathbb{R} \to \mathbb{R}^2$ such that the range $f(\mathbb{R})$ is a closed subset of $\mathbb{R}^2$.

proof Assume such a function exists, and let $B = f(\mathbb{R})$. B is a closed bounded subset of $\mathbb{R}^2$, thus is compact. Then $\tilde{f}$ defined by

$$\tilde{f} : \mathbb{R} \to B$$
$$x \mapsto f(x)$$

is a homeomorphism. Now consider the sequence (x_n) defined by $x_n = n$. Then $(f(x_n))$ is a sequence of elements of the compact set B, therefore admits a converging subsequence $(f(x_{\phi(n)}))_n$. Let y be its limit. Then by (a), $(x_{\phi(n)})$ must converge, and its limit must be $f^{-1}(y)$. However, $(x_{\phi(n)})$ cannot converge since it is unbounded. This leads to a contradiction and proves the result.

(3.5) Let $E \subset \mathbb{R}^n$ be uncountable. Then there exists $x \in \mathbb{R}^n$ such that for any open ball $B(x,r)$, $B(x,r) \cap E$ is uncountable.

proof by contrapositive: assume that for all $x \in \mathbb{R}^n$, there exists $r_x > 0$ such that $B(r_x, x) \cap E$ is countable. Let $B_x = B(r_x, x) \cap E$.

Now consider the metric space (E, d'), where d' is the restriction of the metric d to E. Then the open sets of (E, d') are the intersections of the open sets of $(\mathbb{R}^n, d)$ with E. In particular, $B_x = B(r_x, x) \cap E$ is open in (E, d') for every x. Therefore $\{B_x\}_{x \in E}$ is an open cover of E (by definition of B_x, we have $B_x \subseteq E$, therefore $\cup_{x \in E} B_x \subseteq E$. Conversely, for all $x_0 \in E$, $x_0 \in B_{x_0}$, thus $x_0 \in \cup_{x \in E} B_x$, therefore $E \subseteq \cup_{x \in E} B_x$).

Now since E is separable ($E \cap \mathbb{Q}^n$ is countable and dense in E), every open cover has a countable subcover. Let $\{B_{x_n}\}_{n \in \mathbb{N}}$ be such a subcover. Then we have $E = \cup_{n \in \mathbb{N}} B_{x_n}$, thus E is a countable union of countable sets (by assumption, B_x is countable for every x), therefore E is countable (from problem set 1, a countable union of countable sets is countable).

(3.6) Let f be the uniform limit of the functions $f_n : [0,1] \to [0,1]^2$ defined inductively by the Hilbert construction.

Then we have $\forall x \in [0,1]$

$$\|f_n(x) - f_{n+m}(x)\|_2 \le \sum_{k=0}^{m-1} \|f_{n+k}(x) - f_{n+k+1}(x)\|_2$$

and the distance between $\|f_{n+k}(x) - f_{n+k+1}(x)\|_2$ is at most the diagonal of the square at the $n+k$-th step, i.e. $\|f_{n+k}(x) - f_{n+k+1}(x)\|_2 \le \sqrt{2}(1/2)^{n+k}$. Thus

$$\|f_n(x) - f_{n+m}(x)\|_2 \le \sqrt{2}(1/2)^n \sum_{k=0}^{m-1} (1/2)^k \le \sqrt{2}(1/2)^n (1/2)$$

thus

$$\|f_n - f_{n+m}\|_\infty \le \sqrt{2}(1/2)^n (1/2)$$

Therefore (f_n) is a Cauchy sequence in the complete subspace of continuous functions on $[0,1]^2$ (with the metric induced by the infinite norm), thus converges. Let f be its limit. Then f is continuous (as a uniform limit of continuous functions) and is surjective: let $y \in [0,1]^2$, and fix $\epsilon > 0$. Then we can find x_ϵ such that $\|f(x_\epsilon) - y\| \le \epsilon$: there exists $N_1 \in \mathbb{N}$ and $x_\epsilon \in [0,1]$ such that for all $n \ge N_1$, $\|f_n(x_\epsilon) - y\| \le \epsilon/2$. And since (f_n) converges uniformly to f, there exists N_2 such that for all $n \ge N_2$, $\|f - f_n\|_\infty \le \epsilon/2$. Let $N = \max(N_1, N_2)$, then $\|f(x) - y\| \le \|f - f_N\|_\infty + \|f_N(x) - y\| \le \epsilon$.

Now construct a sequence (x_n) of elements of $[0,1]$ such that $\forall n$ $\|f(x_n) - y\| \le 1/n$. Since $[0,1]$ is compact, (x_n) admits a converging subsequence $(x_{\phi n})$, let x be its limit. Then $f(x_{\phi(n)})$ is converging by continuity of f, and its limit is $f(x)$ by continuity, and y by construction, therefore $f(x) = y$ and f is surjective.

(3.7) Let $\mathcal{O}$ be a nonempty open subset of $\mathbb{R}$. Then there exists a countable collection $\{\mathcal{I}_j\}$ of pairwise disjoint open intervals such that $\mathcal{O} = \cup_j \mathcal{I}_j$.

proof For every $x \in \mathcal{O}$ let $\mathcal{C}_x = \{\mathcal{J} \subset \mathcal{O} : \mathcal{J} \text{ is an open interval containing } x\}$, and $\mathcal{I}_x = \cup_{\mathcal{J} \in \mathcal{C}_x} \mathcal{J}$. $\mathcal{I}_x$ is by construction an open interval, and a subset of $\mathcal{O}$ that contains x.

We have $\{\mathcal{I}_x\}_{x \in \mathcal{O}}$ is an open cover of $\mathcal{O}$ since

- $\forall x \in \mathcal{O}$, $\forall \mathcal{J} \in \mathcal{C}_x$, $\mathcal{J} \subset \mathcal{O}$, thus $\cup_{\mathcal{J} \in \mathcal{C}_x} \mathcal{J} \subseteq \mathcal{O}$, i.e. $\mathcal{I}_x \subseteq \mathcal{O}$. Thus $\cup_{x \in \mathcal{O}} \mathcal{I}_x \subseteq \mathcal{O}$.
- $\forall x \in \mathcal{O}$, let $x \in \mathcal{I}_x$, thus $x \in \cup_{x \in \mathcal{O}} \mathcal{I}_x$. Therefore $\mathcal{O} \subseteq \cup_{x \in \mathcal{O}} \mathcal{I}_x$. This proves that $\mathcal{O} = \cup_{x \in \mathcal{O}} \mathcal{I}_x$

The distinct elements of $\{\mathcal{I}_x\}_{x \in \mathcal{O}}$ are disjoint: to prove this (by contrapositive), assume that $\mathcal{I}_x \cap \mathcal{I}_{x'} \neq \emptyset$. Then $\mathcal{I}_x \cup \mathcal{I}_{x'}$ is an open interval, is a subset of $\mathcal{O}$, and contains x, thus it is a member of C_x, thus $\mathcal{I}_x \cap \mathcal{I}_{x'} \subseteq \mathcal{I}_x$.

Obviously, we also have $\mathcal{I}_x \subseteq \mathcal{I}_x \cap \mathcal{I}_{x'}$, therefore we have $\mathcal{I}_x = \mathcal{I}_x \cap \mathcal{I}_{x'}$. Similarly, we have $\mathcal{I}_{x'} = \mathcal{I}_x \cap \mathcal{I}_{x'}$, which proves the result.

Finally, $\{\mathcal{I}_x\}_{x\in\mathcal{O}}$ is countable, since every distinct element $\mathcal{I}_x$ contains a rational, say $r(\mathcal{I}_x) \in \mathcal{I}_x$ (since $\mathbb{Q}$ is dense in $\mathbb{R}$), one can construct a map (using the axiom of choice)

$$\begin{aligned} i : \{\mathcal{I}_x\}_{x\in\mathcal{O}} &\to \mathbb{Q} \\ \mathcal{I}_x &\mapsto r(\mathcal{I}_x) \end{aligned}$$

i is injective (since the elements of $\{\mathcal{I}_x\}_{x\in\mathcal{O}}$ are disjoint, and $r(\mathcal{I}_x) \in \mathcal{I}_x$), and $\mathbb{Q}$ is countable, therefore $\{\mathcal{I}_x\}_{x\in\mathcal{O}}$ is countable.