MATH 202A - Problem Set 13

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(13.1) Let $y \in \mathbb{R}$. Then

1. E is a Borel set if and only if $y + E$ is a Borel set
2. E is Lebesgue measurable if and only if $y + E$ is Lebesgue measurable.

proof

1. Let $z \in \mathbb{R}$, and consider the function
$$\begin{aligned} f_z : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x + z \end{aligned}$$
We have f_z is Borel-measurable since for all $c \in \mathbb{R}$, $\{x \in \mathbb{R} | f_z(x) \leq c\} = \{x \in \mathbb{R} | x + z \leq c\} = (-\infty, c - z) \in \mathcal{B}$. We also observe that if $E \subseteq \mathbb{R}$, then $f_z^{-1}(E) = \{x \in \mathbb{R} | x + z \in E\} = \{-z + (x + z) | x + z \in E\} = -z + E$. Thus we have: if E is a Borel set, then $f_{-y}^{-1}(E) = y + E$ is a Borel set. Conversely, if $y + E$ is a Borel set, then $-y + (y + E) = E$ is a Borel set. This proves the equivalence.

2. Let E be a Lebesgue measurable set. Then $E = E' \Delta Z$ for some $E' \in \mathcal{B}$ and $Z \subset Z' \in \mathcal{B}$ with $m(Z') = 0$. Then we have
$$\begin{aligned} y + E &= f_{-y}^{-1} E \\ &= f_{-y}^{-1}(E' \Delta Z) \\ &= f_{-y}^{-1}(E') \Delta f_{-y}^{-1}(Z) && \text{since inverse images preserve set operations} \\ &= (y + E') \Delta (y + Z) \end{aligned}$$
where $y + E' \in \mathcal{B}$, and $y + Z \subseteq y + Z' \in \mathcal{B}$, and $m(y + Z') = m(Z') = 0$. Therefore $y + E$ is Lebesgue measurable.
Conversely, if $y + E$ is Lebesgue measurable, then $-y + (y + E) = E$ is Lebesgue measurable. This proves the equivalence.

(13.2) Let $E \subset \mathbb{R}$ be a Borel set. Then the function
$$\begin{aligned} F : \mathbb{R} &\to \mathbb{R}^* \\ t &\mapsto F(t) = m(E \cap (-\infty, t]) \end{aligned}$$
is continuous on $\mathbb{R}$.

proof Let $t \in \mathbb{R}$, and let (t_n) be any sequence of reals that converges to t. Then we have for all $n \in \mathbb{N}$, if $t_n \leq t$, then $(-\infty, t] = (-\infty, t_n] \cup (t_n, t)$ disjointly, then taking the measure, we have $F(t) = F(t_n) + |t - t_n|$. Similarly, if $t \leq t_n$, then $(-\infty, t_n] = (-\infty, t] \cup (t, t_n)$ disjointly, thus $F(t_n) = F(t) + |t_n - t|$. Therefore we have in both cases
$$F(t_n) = F(t) + \epsilon_n$$
where $|\epsilon_n| = |t_n - t|$. We have $\epsilon_n \to 0$, thus $(F(t_n))_n$ converges (in $\mathbb{R}^*$), and $\lim_{n\to\infty} F(t_n) = F(t)$. This proves that F is continuous at t. Since this holds for any t, F is continuous on $\mathbb{R}$.

(13.3) Let E be a Lebesgue measurable set such that $m(E) > 0$. Then E contains a subset which is not Lebesgue measurable.

proof Suppose, by contradiction, that all subset of E are Lebesgue measurable.

Consider the equivalence relation on $\mathbb{R} \times \mathbb{R}$: $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Then let $\mathcal{E}$ be a set that contains one and only one element of each equivalence class[1]. $\mathcal{E}$ satisfies the following properties:

1. for all $y \in \mathbb{R}$, there exists $x \in \mathcal{E}$ such that $x - y \in \mathbb{Q}$, i.e. there exists $x \in \mathcal{E}$ and $q \in \mathbb{Q}$ such that $y = q + x$. Therefore
$$\mathbb{R} = \cup_{q \in \mathbb{Q}} (q + \mathcal{E})$$
2. $x, x' \in \mathcal{E} \Rightarrow x - x' \notin \mathbb{Q}$. Thus if p, q are distinct rationals, then $p + \mathcal{E}$ and $q + \mathcal{E}$ are disjoint. Indeed, if $x \in p + \mathcal{E}$ and $x' \in q + \mathcal{E}$, then $x - p$ and $x' - q$ are elements of $\mathcal{E}$, thus $x - p - (x') - q \notin \mathbb{Q}$, thus $x - x' \notin \mathbb{Q}$, in particular, $x - x' \neq 0$.

Now we have $E = \mathbb{R} \cap E = (\cup_{q \in \mathbb{Q}} (q + \mathcal{E})) \cap E = \cup_{q \in \mathbb{Q}} ((q + \mathcal{E}) \cap E)$, and the union is disjoint. Therefore by σ-additivity of the Lebesgue measure m, we have

$$m(E) = \sum_{q \in \mathbb{Q}} m((q + \mathcal{E}) \cap E)$$

and since $m(E) > 0$, there exists $q_0 \in \mathbb{Q}$ such that $m(\mathcal{E}_0 \cap E) > 0$ where $\mathcal{E}_0 = q_0 + \mathcal{E}$. Next, we have $\mathbb{R} = \cup_{n \in \mathbb{Z}} [n, n+1)$ disjointly, thus taking the intersection with $\mathcal{E}_0 \cap E$ and using σ-additivity of m, we have

$$m(E) = \sum_{n \in \mathbb{Z}} m([n, n+1) \cap \mathcal{E}_0 \cap E)$$

and since $m(E) > 0$, there exists $n_0 \in \mathbb{Z}$ such that $m([n_0, n_0 + 1) \cap \mathcal{E}_0 \cap E) > 0$. Finally, we have

$$\cup_{q \in \mathbb{Q} \cap [0,1]} q + ([n_0, n_0 + 1) \cap (q_0 + \mathcal{E}) \cap E) \subseteq [n_0, n_0 + 2)$$

but this union is disjoint by the second property of $\mathcal{E}$, therefore taking the measure, we have

$$\sum_{q \in \mathbb{Q} \cap [0,1]} m(q + [n_0, n_0 + 1) \cap (q_0 + \mathcal{E}) \cap E) \leq 2$$

and using the fact that $m(y + A) = m(A)$ for any measurable subset A, we have that each term in the sum is equal to $m([n_0, n_0 + 1) \cap (q_0 + \mathcal{E}) \cap E)$, and since the sum has an infinite number of terms and $m([n_0, n_0 + 1) \cap (q_0 + \mathcal{E}) \cap E) > 0$, we have

$$+\infty \leq 2$$

which is a contradiction. This proves the result.

[1]Such a set can be obtained as the image of the quotient set $\mathbb{R}_{/\sim}$ by an injection $i : \mathbb{R}_{/\sim} \to \mathbb{R}$, and such an injection exists by the axiom of choice and the fact that the function

$$s : \mathbb{R} \to \mathbb{R}_{/\sim}$$
$$x \mapsto [x]$$

is a surjection.

(13.4) Let $(X, \mathcal{A}, \mu)$ be a measure space, and $(X, \bar{\mathcal{A}}, \bar{\mu})$ its completion. Let $f : X \to \mathbb{R}^*$ be a measurable function with respect to $\bar{\mathcal{A}}$. Then there exists a function $g : X \to \mathbb{R}^*$ measurable with respect to $\bar{\mathcal{A}}$, with $g = f$ $\bar{\mu}$-almost everywhere.

proof For all $q \in \mathbb{Q}$, let

$$E_q = f^{-1}((-\infty, q])$$

since f is measurable with respect to $\bar{\mathcal{A}}$, for all $q \in \mathbb{Q}$, $E_q \in \bar{\mathcal{A}}$, thus there exists $A_q \in \mathcal{A}$ such that $A_q \subseteq E_q$, and $\bar{\mu}(E_q \setminus A_q) = 0$.[2] We observe that $\forall q, q' \in \mathbb{Q}$, $q \leq q' \Rightarrow E_q \subseteq E_{q'}$, and

$$\forall x \in X,\ f(x) = \inf\{q \in \mathbb{Q} | f(x) \leq q\} = \inf\{q \in \mathbb{Q} | x \in E_q\}$$

Now for all $q \in \mathbb{Q}$, define the set

$$\bar{A}_q = \cup_{r \in \mathbb{Q}, r \leq q} A_r$$

Then $\bar{A}_q \in \mathcal{A}$ since it is the union of countably many measurable sets. We also have $\forall q, q' \in \mathbb{Q}$, $q \leq q' \Rightarrow \bar{A}_q \subseteq \bar{A}_{q'}$.

Now we define the function g:

$$\begin{aligned} g : X &\to \mathbb{R}^* \\ x &\mapsto g(x) = \inf\{q \in \mathbb{Q} | x \in \bar{A}_q\} \end{aligned}$$

with the convention $\inf \emptyset = +\infty$. We observe that g can also be written as a pointwise infimum of measurable functions:

$$g(x) = \inf_{q \in \mathbb{Q}} g_q(x)$$

where for all $q \in \mathbb{Q}$, g_q is defined by

$$\begin{aligned} g_q : X &\to \mathbb{R}^* \\ x &\mapsto g_q(x) = \begin{cases} q & \text{if } x \in \bar{A}_q \\ +\infty & \text{otherwise} \end{cases} \end{aligned}$$

g_q is measurable as the sum of two measurable functions, $g_q = q1_{\bar{A}_q} + \infty 1_{\bar{A}_q^c}$, where $\bar{A}_q$ and its complements are elements of $\mathcal{A}$. Therefore g is $\mathcal{A}$-measurable as the pointwise infimum of countably many $\mathcal{A}$-measurable functions.

Finally, we have for all $q \in \mathbb{Q}$,

$$g(x) \leq q \Leftrightarrow q \in \bar{A}_q$$

indeed, we have

- if $g(x) \leq q$, then by definition of the inf, for all $n \geq 1$, there exists $p_n \in [g(x), g(x) + 1/n]$ such that $x \in \bar{A}_{p_n}$. But $p_n \leq g(x) + 1/n \leq q + 1/n$, thus $\bar{A}_{q_n} \subseteq \bar{A}_{q+1/n}$. Therefore for all $n \geq 1$, $x \in \bar{A}_{q+1/n}$, thus

$$x \in \cap_{n \geq 1} \bar{A}_{q+1/n} = \bar{A}_q$$

[2] Indeed, there exist measurable sets E'_q, S'_q, T'_q, subsets $S_q \subseteq S'_q$ and $T_q \subseteq T'_q$, such that $E_q = (E'_q \cup S_q) \setminus T_q$. Then if we let

$$A_q = E'_q \setminus T'_q$$

then we have A_q is measurable, $A_q \subseteq E_q$, and $E_q \setminus A_q \subseteq S'_q \cup T'_q$, thus

$$\begin{aligned} \bar{\mu}(E_q \setminus A_q) &\leq \bar{\mu}(S'_q) + \bar{\mu}(T'_q) \\ &= \mu(S'_q) + \mu(T'_q) \\ &= 0 \end{aligned}$$

- if $x \in \bar{A}_q$, then $q \in \{p \in \mathbb{Q} | x \in \bar{A}_p\}$ therefore taking the inf, $g(x) \leq q$. This proves the claim.

We have for all $q \in \mathbb{Q}$, and for all rational $r \leq q$, $A_r \subseteq A_q \subseteq E_q$, therefore

$$\bar{A}_q = \cup_{r \in \mathbb{Q}, r \leq q} A_r \subseteq E_q$$

therefore we have for all $x \in X$, $\inf\{q \in \mathbb{Q} | x \in E_q\} \leq \inf\{q \in \mathbb{Q} | x \in \bar{A}_q\}$, i.e.

$$f(x) \leq g(x)$$

and we have

$$\begin{aligned} f(x) \neq g(x) &\Rightarrow f(x) < g(x) \\ &\Rightarrow \exists q \in \mathbb{Q} : f(x) \leq q < g(x) \\ &\Rightarrow \exists q \in \mathbb{Q} : x \in E_q \cap \bar{A}_q^c \\ &\Rightarrow x \in \cup_{q \in \mathbb{Q}} E_q \setminus \bar{A}_q \end{aligned}$$

Therefore if $D = \{x \in X | f(x) \neq g(x)\}$, we have

$$\begin{aligned} \bar{\mu}(D) &\leq \bar{\mu}(\cup_{q \in \mathbb{Q}} E_q \setminus \bar{A}_q) \\ &\leq \sum_{q \in \mathbb{Q}} \bar{\mu}(E_q \setminus \bar{A}_q)) \end{aligned}$$

but $E_q \setminus \bar{A}_q \subseteq E_q \setminus A_q$ (since $A_q \subseteq \bar{A}_q$) therefore for all q, $\bar{\mu}(E_q \setminus \bar{A}_q) = 0$. Therefore $\bar{\mu}(D) = 0$, and f and g disagree on a set of measure 0.

(13.5) Let X be a set, and $\mathcal{A} = P(X)$ be the σ-algebra of its subsets. Let $f_1, \ldots, f_N : X \to \mathbb{R}$ be a finite collection of measurable functions, and $h : \mathbb{R}^N \to \mathbb{C}$ be a continuous function. Then the function

$$\begin{aligned} g : X &\to \mathbb{C} \\ x &\mapsto h(f_1(x), \ldots, f_N(x)) \end{aligned}$$

is measurable (i.e. for all open sets $O \subset \mathbb{C}$, $g^{-1}(O)$ is a measurable subset of $\mathbb{R}$)

proof Let F be the function

$$\begin{aligned} F : X &\to \mathbb{R}^n \\ x &\mapsto (f_1(x), \ldots, f_n(x)) \end{aligned}$$

We first prove that F is measurable. We first observe that any open set in $\mathbb{R}^n$ is the union of countably many basic open sets of the form $U_1 \times U_n$, where for all i, U_i is open. Therefore it suffices to show that the inverse image by F of basic open sets. Let $U = U_1 \times \cdots \times U_n$ be a basic open set. Then we have

$$\begin{aligned} x \in F^{-1}(U) &\Leftrightarrow (f_1(x), \ldots, f_N(x)) \in U \\ &\Leftrightarrow f_i(x) \in U_i \forall i \in \{1, \ldots, N\} \\ &\Leftrightarrow x \in \cap_{i=1}^N f_i^{-1}(U_i) \end{aligned}$$

where for all i, $f_i^{-1}(U_i)$ is measurable since f_i is measurable and U_i is open. Therefore F is measurable.

Now let O be an open subset of $\mathbb{C}$. We have $g = h \circ F$, therefore $g^{-1}(O) = F^{-1}(h^{-1}(O))$. Since h is continuous and O is open, we have $h^{-1}(O)$ is open. Then since F was proven to be measurable, $F^{-1}(h^{-1}(O))$ is measurable. This proves that g is measurable.

(13.6) A measure space is said to be nonatomic if for all $E \in \mathcal{A}$ such that $\mu(E) > 0$, there exists a measurable subset $E' \subset E$ such that $0 < \mu(E') < \mu(E)$. (Note: if $\mathcal{A} = P(X)$ and μ is the counting measure, then μ is not nonatomic.)

Let $(X, \mathcal{A}, \mu)$ be a nonatomic measure space, and let $E \in \mathcal{A}$ such that $\mu(E) > 0$. Then for all $t \in [0, \mu(E)]$, there exists a measurable set $F \subseteq E$ such that $\mu(F) = t$.

proof We seek to find a function $s : [0, \mu(E)] \to \mathcal{A} \cap P(E)$ such that $s(t)$ is a measurable subset that satisfies $\mu(s(t)) = t$. To show existence of such a function, we consider the set $\mathcal{S}$ of monotone non decreasing functions $s : D_s \to \mathcal{A}$ (where the ordering on $\mathcal{A}$ is set inclusion) defined on a subset $D_s \subseteq [0, \mu(E)]$, and such that for all $t \in D_s$, $\mu(s(t)) = t$.

For every $s \in \mathcal{S}$, let $\mathcal{G}_s = \{(t, s(t)), t \in D_s\}$ be the graph of s. Then equip $\mathcal{S}$ with the partial ordering $\leq$ defined by :

$$s \leq s' \Leftrightarrow \mathcal{G}_s \subseteq \mathcal{G}_{s'}$$

Then we have:

$\mathcal{S}$ contains the function

$$\begin{aligned} s_0 : \{0\} &\to \mathcal{A} \\ 0 &\mapsto \emptyset \end{aligned}$$

therefore $\mathcal{S}$ is nonempty.

For every chain $C = \{s_\alpha\}_{\alpha \in A}$ of $\mathcal{S}$ (a totally ordered subset of $\mathcal{S}$), the function s given by its graph

$$\mathcal{G}_s = \cup_{\alpha \in A} \mathcal{G}_{s_\alpha}$$

is an element of $\mathcal{S}$ and an upper bound of C:

- the function s is well defined since by definition, the graphs $\{\mathcal{G}_{s_\alpha}\}$ are totally ordered (with inclusion), therefore their union is a graph.
- the domain of s is $D_s = \cup_{\alpha \in A} D_{s_\alpha}$ and is a subset of $[0, \mu(E)]$ since each D_{s_α} is a subset of $[0, \mu(E)]$.
- for all $t \in D_s$, there exists $\alpha \in A$ such that $t \in D_{s_\alpha}$ (since $D_s = \cup_{\alpha \in A} D_{s_\alpha}$), and we have

  $$\mu(s(t)) = \mu(s_\alpha(t)) = t$$

- for all $t, t' \in D_s$ such that $t \leq t'$, there exist $\alpha, \alpha' \in A$ such that $t \in D_{s_\alpha}$ and $t' \in D_{s_{\alpha'}}$. Since the chain is totally ordered, we have either $s_\alpha \leq s_{\alpha'}$ or $s_\alpha \leq s_{\alpha'}$. Assume without loss of generality that $s_{\alpha'} \leq s_\alpha$. Then $D_{s_{\alpha'}} \subseteq D_{s_\alpha}$ and we have t, t' are both in D_{s_α}. Thus $s_\alpha(t) \subseteq s_\alpha(t')$ (since s_α is monotone non decreasing), but $s(t) = s_\alpha(t)$ and $s(t') = s_\alpha(t')$, therefore

  $$s(t) \leq s(t')$$

  and s is monotone non decreasing.

Therefore s is an element of $\mathcal{S}$. It is, by definition, an upper bound on the chain C.

Therefore by Zorn's Lemma, $\mathcal{S}$ has a maximal element $s^{\max}$. Next, we show that the domain of $s^{\max}$, $D_{s^{\max}}$ is $[0, \mu(E)]$. Suppose by contradiction that this is not the case. Let $D^c_{s^{\max}} = [0, \mu(E)] \setminus D_{s^{\max}}$, and consider two cases:

1. if $D^c_{s^{\max}}$ is relatively open in $[0, \mu(E)]$, then it is a union of disjoint open intervals (in $[0, \mu(E)]$). Let $t_1 < t_2$ be the boundaries of one such open interval. Then we have $t_1, t_2 \in D_{s^{\max}}$. Let $F_1 = s^{\max}(t_1)$ and $F_2 = s^{\max}(t_2)$. Then we have $F_1 \subseteq F_2$, and they are both measurable subsets of E. Now we have $F_2 \setminus F_1$ is measurable with measure

   $$\begin{aligned} \mu(F_2 \setminus F_1) &= \mu(F_2) - \mu(F_1) \\ &= t_2 - t_1 > 0 \end{aligned}$$

therefore, since the space is nonatomic, there exists a measurable F subset of $F_2 \setminus F_1$ with measure $\mu(F) \in (0, t_2 - t_1)$. Then we have $F_1 \cup F$ is measurable, is a subset of E, and has measure

$$\begin{aligned} \mu(F_1 \cup F) &= \mu(F_1) + \mu(F) && \text{since } F_1 \text{ and } F \text{ are disjoint} \\ &= t_1 + \mu(F) \\ &\in (t_1, t_2) \end{aligned}$$

let $t_0 = t_1 + \mu(F)$. Then we can extend the function $s^{\max}$ as follows

$$\bar{s} : D_{s^{\max}} \cup \{t_0\} \to \mathcal{A}$$

where $\bar{s}$ coincides with s on $D_{s^{\max}}$, and $\bar{s}(t_0) = F_1 \cup F$. Then $\bar{s}$ is still an element of $\mathcal{S}$, and $s^{\max} < \bar{s}$, which contradicts maximality of $s^{\max}$

2. if $D^c_{s^{\max}}$ is not relatively open in $[0, \mu(E)]$, then there exists $t_0 \in D^c_{s^{\max}}$ such that for all $\epsilon > 0$, $(t_0 - \epsilon, t_0 + \epsilon)$ has nonempty intersection with $D_{s^{\max}}$. Then construct a sequence (t_n) of elements of $D_{s^{\max}}$ such that $(|t_n - t_0|)_{n \geq 1}$ is non-increasing and converges to 0. Then we can extract a subsequence of elements that are on one side of t_0 (this is possible since at least one side has to contain an infinite number of elements). Therefore, assume that $(t_n)_{n \geq 1}$ is a sequence of elements of $D_{s^{\max}}$, such that $t_n \leq t_0$ for all n (the case $t_n \geq t_0$ is treated similarly). In particular (t_n) is non-decreasing and converges to t_0.

   Now let $F_n = s^{\max}(t_n)$. Then we have for all n, $t_n \leq t_{n+1}$, thus $F_n \subseteq F_{n+1} \subseteq E$. Therefore we have $F = \cup_{n \geq 1} F_n$ is a measurable subset of E, and by σ-additivity of m,

$$\mu(\cup_n F_n) = \lim_{n \to \infty} \mu(F_n) = \lim_{n \to \infty} t_n = t_0$$

   therefore we can extend $s^{\max}$ by defining $\bar{s}(t_0) = \mu(\cup_{n \geq 1} F_n)$. Then $\bar{s}$ is still an element of $\mathcal{S}$ and $s^{\max} < \bar{s}$, which contradicts maximality of $s^{\max}$.

Both cases lead to a contradiction, therefore $D_{s^{\max}} = [0, \mu(E)]$.
Finally, this provides the desired function

$$s : [0, \mu(E)] \to \mathcal{A} \cap P(E)$$

which satisfies for all $t \in [0, \mu(E)]$, $F = s(t)$ is a measurable subset of E such that $\mu(F) = \mu(s(t)) = t$.

(13.7) Product measures. Let $(\mathbb{R}, \mathcal{B}, m)$ be the real line with Lebesgue measure on the Borel sets. Let $(X, \mathcal{A}, \mu)$ be any measure space. Let $f : X \to [0, \infty)$ be measurable with respect to $\mathcal{A}$, and consider the graph G of f, and the region $\mathcal{R}$ under the graph of f:

$$\mathcal{R} = \{(x,t) \in X \times \mathbb{R} | 0 \leq t < f(x)\}$$
$$\mathcal{G} = \{(x,t) \in X \times \mathbb{R} | t = f(x)\}$$

Then

1. $\mathcal{R}$ is a measurable subset of $(X \times \mathcal{R}, \mathcal{A} \times \mathcal{B})$
2. The measure of $\mathcal{R}$ is $\mu \times m(\mathcal{R}) = \int_X f d\mu$
3. $\mathcal{G}$ is measurable (in $\mathcal{A} \times \mathcal{B}$), and if μ is σ-finite then $(\mu \times m)(\mathcal{G}) = 0$ (thus with suitable interpretations, the integral of f is the area under its graph)

proof

1. We have

$$\begin{aligned}(x,t) \in \mathcal{R} &\Leftrightarrow 0 \leq t < f(x)\\ &\Leftrightarrow \exists q \in \mathbb{Q} : 0 \leq t \leq q < f(x)\\ &\Leftrightarrow \exists q \in \mathbb{Q} : (x,t) \in f^{-1}((q, +\infty)) \times [0, q]\end{aligned}$$

   Therefore
$$\mathcal{R} = \cup_{q \in \mathbb{Q}} f^{-1}(q, +\infty) \times [0, q]$$
   and for all $q \in \mathbb{Q}$, $f^{-1}((q, +\infty)) \times [0, q]$ is measurable in the product space. Therefore $\mathcal{R}$ is the union of countably many measurable sets and is measurable.

2. First, consider the case where $f = s$ is a non-negative simple function. Then there exists a partition of X into a finite number of disjoint measurable sets $X = \cup_{i=1}^k A_i$ and reals $\alpha_1, \ldots, \alpha_k$ such that

$$s = \sum_{i=1}^{k} \alpha_k 1_{A_k}$$

   and we have for all i,

$$\begin{aligned}\mathcal{R} \cap A_i \times \mathbb{R} &= \{(x,t) \in A_i \times \mathbb{R} | 0 \leq t < s(x)\}\\ &= \{(x,t) \in A_i \times \mathbb{R} | 0 \leq t < \alpha_i\}\\ &= A_i \times [0, \alpha_i)\end{aligned}$$

   Therefore we have the partition
$$\mathcal{R} = \cup_{i=1}^k A_i \times [0, \alpha_i)$$
   where each element of the partition is measurable. Using additivity of $\mu \times m$,

$$\begin{aligned}(\mu \times m)(\mathcal{R}) &= \sum_{i=1}^{k} (\mu \times m)(A_i \times [0, \alpha_i))\\ &= \sum_{i=1}^{k} \mu(A_i) m([0, \alpha_i))\\ &= \sum_{i=1}^{k} \alpha_i \mu(A_i)\\ &= \int_X s d\mu\end{aligned}$$

This proves the result for simple functions.

Now consider the general case. Let $f : X \to [0, \infty)$ be measurable, and let $(s_n)_{n\in\mathbb{N}}$ be a sequence of non-negative simple functions that converges to f, and such that for all n, $0 \le s_n \le s_{s+1} \le f$. For all n, let $\mathcal{R}_n$ be the region under the graph of s_n. Then we have for all n, if $(x,t) \in \mathcal{R}_n$, then $0 \le t < s_n(x)$, and since $s_n(x) \le s_{n+1}(x)$, we also have $0 \le t < s_{n+1}(x)$, i.e. $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$. We also have

$$\mathcal{R} = \cup_{n\ge 1}\mathcal{R}_n$$

indeed, if $(x,t) \in \mathcal{R}_n$ for some n, then $0 \le t < s_n(x)$, but $s_n(x) \le f(x)$, thus $(x,t) \in \mathcal{R}$. Conversely, if $(x,t) \in \mathcal{R}$, then $0 \le t < f(x)$, and since $s_n(x) \to f(x)$, there exists n such that $0 \le t < s_n(x) \le f(x)$, i.e. $(x,t) \in \mathcal{R}_n$.

Finally, we have by the monotone converges theorem and σ-additivity of $\mu \times m$

$$\begin{aligned}
\int_X f d\mu &= \lim_{n\to\infty} \int_X s_n d\mu \\
&= \lim_{n\to\infty} (\mu \times m)\mathcal{R}_n \\
&= (\mu \times m) \cup_{n\in\mathbb{N}} \mathcal{R}_n \\
&= (\mu \times m)\mathcal{R}
\end{aligned}$$

3. $\mathcal{G}$ is measurable: for all $n \ge 1$ let $\mathcal{G}_n = \{(x,t) \in X \in f(x) - 1/n < t < f(x) + 1/n\}$. Then we have

$$\begin{aligned}
(x,t) \in \mathcal{G} &\Leftrightarrow f(x) = t \\
&\Leftrightarrow \forall n \ge 1,\ f(x) - 1/n < t < f(x) + 1/n \\
&\Leftrightarrow (x,t) \in \cap_{n=1}^{\infty}\mathcal{G}_n
\end{aligned}$$

and we have for all n

$$\begin{aligned}
(x,t) \in \mathcal{G}_n &\Leftrightarrow f(x) - 1/n < t < f(x) + 1/n \\
&\Leftrightarrow \exists p, q \in \mathbb{Q} : f(x) - 1/n < p < t < q < f(x) + 1/n \\
&\Leftrightarrow \exists p, q \in \mathbb{Q} : f(x) \in (q - 1/n, p + 1/n) \text{ and } t \in (p,q) \\
&\Leftrightarrow (x,t) \in \cup_{p,q\in\mathbb{Q}} \left(f^{-1}((q - 1/n, p + 1/n)) \times (q,p)\right)
\end{aligned}$$

therefore $\mathcal{G}_n$ is the union of countably many measurable sets, and is measurable. It follows that $\mathcal{G}$ is measurable.

Finally, by the Fubini theorem, we have

$$\begin{aligned}
(\mu \times m)(\mathcal{G}) &= \int_{X\times\mathbb{R}} 1_{\mathcal{G}}(x,t) d(\mu \times m) \\
&= \int_X \left(\int_{\mathbb{R}} 1_{\mathcal{G}}(x,t) dm(t) \right) d\mu(x)
\end{aligned}$$

but for all $x \in X$, $1_{\mathcal{G}}(x,t) = 1_{\{f(x)\}}(t)$, thus $\int_{\mathbb{R}} 1_{\mathcal{G}}(x,t)dm(t) = \int_{\mathbb{R}} 1_{\{f(x)\}}dm(t) = m(\{f(x)\}) = 0$. Therefore

$$(\mu \times m)(\mathcal{G}) = \int_X 0 d\mu(x) = 0$$