MATH 202A - Problem Set 11

Walid Krichene (23265217)

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(11.1) Consider the set $X = \{1, 2, 3\}$, and the σ-algebra $\mathcal{A} = P(X)$, and define the set function

$$\rho : P(X) \to \mathbb{R}$$

such that $\rho(\emptyset) = 0$, $\rho(\{1\}) = -2$, $\rho(\{2\}) = 4$, $\rho(\{3\}) = -3$, $\rho(\{1,2\}) = 2$, $\rho(\{1,3\}) = -5$, $\rho(\{2,3\}) = 1$, $\rho(1, 2, 3) = -1$. Then ρ is additive, and the sets $A = \{1, 2\}$ and $B = \{2, 3\}$ satisfy

$$\rho(A) = 2 > 0$$
$$\rho(B) = 1 > 0$$
$$\rho(A \cup B) = -1 < 0$$

(11.2) Let $\mathbb{Q} = \{q_k, k \in \mathbb{N}\}$. Let $\mathcal{O}_n = \cup_{k\in\mathbb{N}}(q_k - \frac{1}{n2^k}, q_k + \frac{1}{n2^k})$. Let $E = \cap_{n=1}^{\infty}\mathcal{O}_n$. Then $E \subset \mathbb{R}$ is Borel set, E is of the second category (i.e. not of the first category), and $m(E) = 0$ where m is the Lebesgue measure.

proof E is a Borel set: for each $n \in \mathbb{N}$, $\mathcal{O}_n$ is open as the union of open sets. Therefore E is the countable intersection of open sets, thus is a Borel set.

E is of the second category: we first prove that O_n^c is nowhere dense. Indeed, we have for all $q_k \in \mathbb{Q}$, $B(q_k, \frac{1}{n2^k}) \subset \mathcal{O}_n$, thus $q_k \in \text{int}(O_n)$. Therefore $\mathbb{Q} \subseteq \text{int}(O_n)$, therefore $\mathbb{R} = \text{cl}(\mathbb{Q}) \subseteq \text{cl}(\text{int}(O_n))$, i.e. $\text{cl}(\text{int}(O_n)) = \mathbb{R}$. Taking complements, and using the fact that for any set X, $\text{int}(X)^c = \text{cl}(X^c)$ and $\text{cl}(X)^c = \text{int}(X^c)$, we have

$$\text{int}(\text{cl}(O_n^c)) = \emptyset$$

i.e. O_n^c is nowhere dense. Now we have

$$\begin{aligned}\mathbb{R} &= E \cup E^c \\ &= E \cup (\cup_{n=1}^{\infty} O_n^c)\end{aligned}$$

therefore E is not of the first category, otherwise we would have $\mathbb{R}$ is of the first category (countable union of nowhere dense sets), which contradicts Baire's theorem.

$m(E) = 0$. We prove that for all $\epsilon > 0$, $m(E) \le \epsilon$. Let $\epsilon > 0$, and let $n_0 \in \mathbb{N}$ such that $4/n_0 \le \epsilon$. Then we have

$$\begin{aligned}m(E) &\le m(O_{n_0}) && \text{since } E \subseteq O_{n_0} \\ &= m(\cup_{k\ge 0}(q_k - \frac{1}{n_0 2^k}, q_k + \frac{1}{n_0 2^k})) \\ &\le \sum_{k\ge 0} m((q_k - \frac{1}{n_0 2^k}, q_k + \frac{1}{n_0 2^k})) && \text{by } \sigma\text{-additivity of } m \\ &= \sum_{k\ge 0} \frac{2}{n_0 2^k} \\ &= \frac{4}{n_0} \le \epsilon\end{aligned}$$

this proves that $m(E) < \epsilon$ for all $\epsilon > 0$, therefore $m(E) = 0$.

(11.3) Consider the cantor set C constructed as follows: given a sequence of lengths $(r_n)_{n\geq 1}$, $r_n \in (0,1)$, construct a sequence C_n inductively, such that each C_n is the disjoint union of 2^n closed intervals:

- $C_0 = [0,1]$ is the union of 2^0 closed intervals.
- for all $n \geq 0$, assume C_n is constructed, and is the disjoint union of 2^n closed intervals

  $$C_n = \cup_{k=1}^{2^n} I_{n,k}$$

  Then construct C_{n+1} by removing, from each interval $I_{n,k} \subset C_n$, an open subinterval in the middle, of length $r_{n+1}m(I_{n,k})$, forming two intervals. Then C_{n+1} is the disjoint union of 2^{n+1} closed intervals.

Define C to be the intersection $C = \cap_{n=0}^{\infty} C_n$ (i.e. the limit of the decreasing sequence $(C_n)_n$). Then C is a Borel set, $m(C) = \prod_{n=1}^{\infty}(1 - r_n)$, and for all $x \in \mathbb{R}$, the indicator function 1_C is continuous at x if and only if $x \notin C$.

proof C is a Borel set: we have for all $n \geq 0$, C_n is a Borel set (by induction: $C_0 = [0,1]$ is a Borel set, and if C_n is a Borel set, then C_{n+1} can be written $C_{n+1} = C_n \setminus \cup_{k=1}^{2^n} I_{n,k}$ where $I_{n,k}$ is an open interval, thus C_{n+1} is a Borel set). Therefore C is the intersection of countably many Borel sets, and is a Borel set.

The measure of C: by induction on $n \geq 1$, we have $m(C_n) = \prod_{i=1}^{n}(1 - r_i)$. This is true for $n = 1$: $m(C_0) = m([0,(1-r_1)/2] \cup [(1+r_1)/2, 1]) = 1 - r_1$. If it is true for n, then $m(C_{n+1}) = (1 - r_{n+1})m(C_n) = (1 - r_{n+1})\prod_{i=1}^{n}(1 - r_i) = \prod_{i=1}^{n+1}(1 - r_i)$.

Therefore (C_n) is a sequence of decreasing measurable (Borel) sets, which are subsets of $[0,1]$, that has finite measure. By σ-additivity of m, we have

$$\begin{aligned} m(\cap_{n\in\mathbb{N}} C_n) &= \lim_{n\to\infty} m(C_n) \\ &= \lim_{n\to\infty} \prod_{i=1}^{n}(1 - r_i) \\ &= \prod_{i=1}^{\infty}(1 - r_i) \end{aligned}$$

Let $x \notin C$, and let $\epsilon > 0$. Since $x \notin C$, there exists $n \geq 0$ such that $x \notin C_n$, i.e. $x \in C_n^c$. But since C_n is closed (finite union of closed intervals), C_n^c is open, and there exists an open ball $B(x,\delta) \subset C_n^c$. And we have for all $y \in B(x,\delta)$, $y \notin C$, thus $|1_C(y) - 1_C(x)| = 0 \leq \epsilon$. Therefore 1_C is continuous at x.

Now let $x \in C$, let $\epsilon < 1$. We show that for any $\delta > 0$, there exists $y \in B(x,\delta) \cap C^c$ such that $|1_C(y) - 1_C(x)| = |0 - 1| = 1 > \epsilon$, which will prove that 1_C is not continuous at x. To prove that there exists such a $y \in B(x,\delta) \cap C^c$, assume by contradiction that $B(x,\delta) \subseteq C$. Then for all $n \in \mathbb{N}$, $B(x,\delta) \subseteq C_n$. Since C_n is the disjoint union of closed intervals $I_{n,k}$, we must have $B(x,\delta) \subseteq I_{n,k}$ for some k, thus $m(B(x,\delta)) \leq m(I_{n,k})$, i.e. $\delta \leq m(I_{n,k}) \leq 1/2^n$ (proved by induction). But this cannot be true for all n since $\lim_{n\to\infty} 1/2^n = 0$, and we obtain a contradiction.

(11.4) Borel-Cantelli: Let $(X, \mathcal{A}, \mu)$ be a measure space. For $n \in \mathbb{N}$, let $A_n \in \mathcal{A}$, and suppose that $\sum_n \mu(A_n) < \infty$. Let E be the set of points x that belong to infinitely many A_n. Then E is measurable and $\mu(E) = 0$.

proof E is measurable: for any $x \in X$, let $N(x) = \{n \in \mathbb{N} : x \in A_n\}$. We first show that

$$E = \cap_{k=0}^{\infty} \cup_{n=k}^{\infty} A_n$$

indeed, if $x \in E$, then $N(x)$ is an infinite set, thus for all $k \in \mathbb{N}$, there exists $n \geq k$ such that $x \in A_n$ (otherwise $\max N(x) \leq k$ and $N(x)$ would be finite), therefore $x \in \cup_{n \geq k} A_n$, which proves that $E \subseteq \cap_{k=0}^{\infty} \cup_{n=k}^{\infty} A_n$. Conversely, if $x \in \cap_{k=0}^{\infty} \cup_{n=k}^{\infty} A_n$, then for all k, there exists $n \geq k$ such that $x \in A_n$, thus $N(x)$ is infinite ($N(x)$ is nonempty since $x \in \cup_{n=0}^{\infty} A_n$, and it cannot be finite, otherwise it would have a maximum $k = \max N(x)$, which contradicts the fact that there exists $n \geq k+1$ such that $x \in A_n$).

Therefore $E = \cap_{k=0}^{\infty} \cup_{n=k}^{\infty} A_n$ is measurable (since for all n, $A_n \in \mathcal{A}$ and $\mathcal{A}$ is closed under countable intersection and union).

Measure of E: for all $n \in \mathbb{N}$, let $f_n = 1_{A_n}$. Let

$$\begin{aligned} f : X &\to \mathbb{R}^* \\ x &\mapsto \sum_{n \in \mathbb{N}} f_n(x) = \sum_{n \in \mathbb{N}} 1_{A_n}(x) \end{aligned}$$

we have $f(x)$ is the cardinality of $N(x)$, therefore $x \in E \Leftrightarrow f(x) = \infty$.

Now we observe that $f(x) = \lim_{n \to \infty} g_n(x)$ where $g_n(x) = \sum_{k=0}^{n} f_n(x)$ satisfies:

- for all $n \in \mathbb{N}$, (g_n) is a finite sum of (non-negative) measurable functions, therefore is measurable.
- for all $n \in \mathbb{N}$, and for all $x \in X$, $g_n(x) \leq g_{n+1}(x)$

therefore by the monotone convergence theorem,

$$\int_X f d\mu = \lim_{n \to \infty} \int_X g_n d\mu$$

then for all $n \in \mathbb{N}$, we have $\int_X g_n d\mu = \int_X \sum_{k=0}^{n} 1_{A_k} d\mu = \sum_{k=0}^{n} \int_X 1_{A_k} d\mu = \sum_{k=0}^{n} \mu(A_k) \leq \sum_k \mu(A_n)$. Taking the limit as $n \to \infty$, we have

$$\int_X f d\mu \leq \sum_k \mu(A_n) < \infty$$

therefore $\int_X f d\mu$. Finally, since f is non-negative and $E \subseteq X$we have

$$\int_E f d\mu \leq \int_X f d\mu < \infty$$

but $\int_E f d\mu = \infty \mu(E)$, therefore we must have $\mu(E) = 0$ for the product to be finite.

(11.5) Consider a measure space $(\mathbb{R}^m, \mathcal{B}, \mu)$ where $\mathcal{B}$ is the Borel σ-algebra, and such that $\mu(K)$ is finite for every bounded subset $K \subset \mathbb{R}^m$. Let E be a compact set and consider $\mathcal{O}_n = \{x : d(x, E) < 1/n\}$. Then $\lim_{n \to \infty} \mu(\mathcal{O}_n) = \mu(E)$.

proof For all $n \geq 1$, $\mathcal{O}_n$ is the inverse image of the open set $(-\infty, 0)$ by the continuous function $x \mapsto d(x, E)$, thus $\mathcal{O}_n$ is open, therefore is measurable. Let $K_n = \mathcal{O}_1 \setminus \mathcal{O}_n$, K_n is measurable as the intersection of the measurable sets $\mathcal{O}_1$ and $\mathcal{O}_n^c$. We have the following properties:

- For all $n \geq 1$, $\mathcal{O}_n \subset \mathcal{O}_1$, thus $\mathcal{O}_1$ is the disjoint union $\mathcal{O}_1 = \mathcal{O}_n \cup K_n$
- We have

  $$\begin{aligned} E &= \cap_{n \geq 1} \mathcal{O}_n \\ &= \cap_{n \geq 1} \mathcal{O}_1 \setminus K_n \\ &= \mathcal{O}_1 \setminus (\cup_{n \geq 1} K_n) \end{aligned}$$

  therefore $\mathcal{O}_1$ is the disjoint union $\mathcal{O}_1 = (\cup_{n \geq 1} K_n) \cup E$.

We have for all $n \in \mathbb{N}$, $\mathcal{O}_{n+1} \subseteq \mathcal{O}_n$, thus $K_n \subseteq K_{n+1}$. Therefore (K_n) is an increasing sequence of measurable sets, and by σ-additivity of μ,

$$\lim_{n\to\infty} \mu(K_n) = \mu(\cup_{n=0}^{\infty} K_n)$$

We have $\cup_{n=0}^{\infty} K_n$ is a subset of the bounded set $\mathcal{O}_1$[1] thus it has finite measure by assumption, therefore the limit $\lim_{n\to\infty} \mu(K_n)$ exists and is finite.

Next, using the disjoint union $\mathcal{O}_1 = \mathcal{O}_n \cup K_n$, we have

$$\mu(\mathcal{O}_n) = \mu(\mathcal{O}_1) - \mu(K_n)$$

where $\mu(\mathcal{O}_1)$ is finite. Therefore $\lim_{n\to\infty} \mu(\mathcal{O}_n)$ exists, and it is equal to $\mu(\mathcal{O}_1) - \mu(\cup_{n=0}^{\infty} K_n)$. Finally, using the disjoint union $\mathcal{O}_1 = E \cup \cup_{n\geq 1} K_n$, we have

$$\mu(E) = \mu(\mathcal{O}_1) - \mu(\cup_{n\geq 1} K_n) = \lim_{n\to\infty} \mu(\mathcal{O}_n)$$

(11.6) Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where μ is the counting measure. Then Fatou's Lemma and the Dominated Convergence Theorem become:

- Fatou's Lemma: if $(a^{(n)})_{n\in\mathbb{N}}$ is a sequence of non-negative sequences, then for every subset $\mathcal{N} \subseteq \mathbb{N}$

$$\sum_{k\in\mathcal{N}} \liminf_n a_k^{(n)} \leq \liminf_n \sum_{k\in\mathcal{N}} a_k^{(n)}$$

  in particular, if for each $k \in \mathbb{N}$, $a_k = \liminf a_k^{(n)}$ exists, then

$$\sum_{k\in\mathcal{N}} a_k \leq \liminf_n \sum_{k\in\mathcal{N}} a_k^{(n)}$$

- Dominated convergence theorem: Let $\mathcal{N} \subseteq \mathbb{N}$. If $(a^{(n)})_{n\in\mathbb{N}}$ is a sequence of sequences such that
  - there exists a sequence $(b_k)_k$ such that the series $\sum_k b_k$ is absolutely convergent and for all $n \in \mathbb{N}$ for all $k \in \mathcal{N}$, $a_k^{(n)} \leq b_k$
  - for all $k \in \mathcal{N}$, $a_k = \lim_{n\to\infty} a_k^{(n)}$ exists

  then $\lim_{n\to\infty} \sum_{k\in\mathcal{N}} a_k^{(n)}$ exists, the series $\sum_k (a_k)$ is absolutely convergent, and

$$\sum_{k\in\mathcal{N}} a_k = \lim_{n\to\infty} \sum_{k\in\mathcal{N}} a_k^{(n)}$$

(11.7) Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$ be a σ-algebra. Let $f : X \to \mathbb{R}$ be a measurable function. Then for any Borel set $E \subset \mathbb{R}$, $f^{-1}(E)$ is measurable.

[1] $\mathcal{O}_1$ is bounded: since E is compact, it is bounded, thus there exists $x_0 \in X$ and $r > 0$ such that $E \subset B(x_0, r)$, then $\mathcal{O}_1 \subset B(x_0, r+1)$ since for all $x \in \mathcal{O}_1$, the function

$$E \to \mathbb{R}$$
$$e \mapsto d(x, e)$$

is continuous on the compact E, therefore it attains its minimum on E, i.e. there exists $p(x) \in E$ such that $d(x, E) = d(x, p(x))$, and we have $d(x, x_0) \leq d(x, p(x)) + d(p(x), x_0) < 1 + r$

proof Let $\mathcal{E} = \{E \in P(\mathbb{R}) : f^{-1}(E) \in \mathcal{A}\}$. Then we have

- $\mathcal{E}$ contains all open subsets of $\mathbb{R}$: let $E \subseteq \mathbb{R}$ be open. Then E is the countable union of disjoint open intervals, $E = \cup_{n\in\mathcal{N}} I_n$ where $\mathcal{N}$ is countable (could be finite). Then we have

$$f^{-1}(E) = f^{-1}(\cup_{n\in\mathcal{N}} I_n) = \cup_{n\in\mathcal{N}} f^{-1}(I_n)$$

  where for all $n \in \mathcal{N}$, $f^{-1}(I_n)$ is measurable since I_n is an interval. Therefore $f^{-1}(E)$ is the countable union of measurable sets, thus is measurable.

- $\mathcal{E}$ is closed under countable union: let $(E_n)_{n\in\mathbb{N}}$ be a countable collection of elements of $\mathcal{E}$, and let $E = \cup_{n\in\mathbb{N}} E_n$. Then we have

$$f^{-1}(E) = f^{-1}(\cup_{n\in\mathbb{N}} E_n) = \cup_{n\in\mathbb{N}} f^{-1}(E_n)$$

  where for all n, $f^{-1}(E_n) \in \mathcal{A}$ since $E_n \in \mathcal{E}$. Therefore $f^{-1}(E)$ is the countable union of measurable sets, and is measurable.

- $\mathcal{E}$ is closed under complements: let $E \in \mathcal{E}$. Then we have

$$f^{-1}(E^c) = (f^{-1}(E))^c$$

  where $f^{-1}(E) \in \mathcal{A}$ since $E \in \mathcal{E}$. Therefore $f^{-1}(E)$ is the complement of a measurable set, and is measurable.

Therefore $\mathcal{E}$ is a σ-algebra that contains all open sets, therefore $\mathcal{E}$ contains the smallest such σ-algebra, i.e. contains the Borel σ-algebra.

(11.8) Let $I \subset \mathbb{R}$ be a compact interval and let $h : I \to \mathbb{R}$ be a continuous function. Let $\Gamma \subset \mathbb{R}^2$ be the graph of h. Since f is continuous, Γ is closed, thus it is Borel-measurable. And we have $m(\Gamma) = 0$, where m is the two-dimensional Lebesgue measure.

proof We show that $m(\Gamma) \le \epsilon$ for all $\epsilon > 0$.

Let $\epsilon > 0$. Let $a = \inf I$, $b = \sup I$, and $\epsilon' = \frac{\epsilon}{2(b-a+1)}$. Since f is continuous on a compact set, it is uniformly continuous, i.e. there exists $\delta \in (0,1)$ such that for all $x, y \in I$, $|x-y| \le \delta \Rightarrow |f(x) - f(y)| \le \epsilon'$.

Now let, for all $j \in \{0, \ldots, I\}$, $a_j = a + j\delta$, where $J = \lceil \frac{b-a}{\delta} \rceil$. Then we have

$$J - 1 < \frac{b-a}{\delta} \le J$$

thus

$$a + (J-1)\delta < b \le a + J\delta$$

i.e.

$$a_{J-1} < b \le a_J$$

Therefore $I = \cup_{j=0}^{J-1} [a_j, a_{j+1}]$, and for all $j \in \{0, \ldots, J-1\}$, $[a_j, a_{j+1}] \cap I$ is non empty. Let $x_j \in [a_j, a_{j+1}] \cap I$, and let $y_j = f(x_j)$. Then we can cover Gamma as follows:

$$\Gamma \subseteq \cup_{j=0}^{J-1} [a_j, a_{j+1}] \times [y_j - \epsilon', y_j + \epsilon']$$

indeed, for all $(x, f(x)) \in \Gamma$, we have $x \in I$, thus $x \in [a_j, a_{j+1}]$ for some $j \in \{0, \ldots, J-1\}$, and since $|x - x_j| \le \delta$, we have $|f(x) - y_j| \le \epsilon'$, thus $(x, f(x)) \in [a_j, a_{j+1}] \times [y_j - \epsilon', y_j + \epsilon']$. Finally, we can bound the measure of Γ:

$$\begin{aligned} m(\Gamma) &\leq \sum_{j=0}^{J+1} m([a_j, a_{j+1}] \times [y_j - \epsilon', y_j + \epsilon']) \\ &\leq \sum_{j=0}^{J+1} 2\delta\epsilon' \\ &= 2J\delta\epsilon' \end{aligned}$$

But $J - 1 < \frac{b-a}{\delta}$, thus $J\delta < b - a + \delta \leq b - a + 1$, thus

$$m(\Gamma) \leq 2(b - a + 1)\epsilon' = \epsilon$$

thus $m(\Gamma) \leq \epsilon$ for all $\epsilon > 0$, i.e. $m(\Gamma) = 0$.